SECONDARY SCHOOL IMPROVEMENT PROGRAMME (SSIP) 2019 MATHEMATICS

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GRADE 12

SUBJECT: MATHEMATICS

TEACHER NOTES

TABLE OF CONTENTS

SESSION NO: 1

TOPIC: SEQUENCES AND SERIES

Teacher Note:
Sequences and series is an exciting part of the curriculum. Make sure the learners know the difference between arithmetic and geometric sequences.
If the sequence or series is arithmetic then: $d = T_2 - T_1 = T_3 - T_2$
If the sequence or series is geometric then: $r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$
If the sequence is quadratic then there is a constant second difference.
Learners tend to struggle with the meaning of *n*, T_n and S_n. Emphasise to them that the value of *n* represents the term's position in the sequence, whereas T_n represents the actual term with position *n*.
Expressing terms as follows is important for learners to understand:
For arithmetic: $T_3 = a + 2d$; $T_4 = a + 3d$
For geometric: $T_3 = ar^2$; $T_4 = ar^3$
Solving simultaneous equations is extremely important and needs to be revised before the start of this chapter. Learners often struggle to understand the meaning of the sum to infinity of a convergent geometric series. Emphasise to them that the sum to infinity is the value approached as terms are added progressively.
Applications of sequences and series in the real world are important for learners to focus on.

LESSON OVERVIEW

1. Introduction session: 5 minutes
2. Typical exam questions and solutions: 60 minutes

SECTION A: TYPICAL EXAM QUESTIONS

QUESTION 1

(a). Consider the sequence -2; 3; 8; 13; 18; 23; 28; 33; 38;……………..
(1) Determine the 100th term. (2)
(2) Determine the sum of the first 100 terms. (2)

(b) The 13th and 7th terms of an arithmetic sequence are 15 and 51 respectively. Which term of the sequence is equal to -21 (6)

QUESTION 2

In a geometric sequence, the 6^{th} term is 243 and the 3^{rd} term is 72.

Determine:

(a) The constant ratio. (4)

(b) The sum of the first 10 terms. (4)

QUESTION 3 (DoE Nov 2008 Paper 1)

Consider the sequence: $\frac{1}{2}\,;4\,;\frac{1}{4}\,;7\,;\frac{1}{8}\,;\ 10\,;\ldots$

(a) If the pattern continues in the same way, write down the next TWO terms in the sequence. (2)

(b) Calculate the sum of the first 50 terms of the sequence. (7)

QUESTION 4

Given the geometric series: $8x^2+4x^3+2x^4+\ldots$

(a) Determine the n^{th} term of the series. (3)

(b) For what value(s) of x will the series converge? (2)

(c) Calculate the sum of the series to infinity if $x=\frac{3}{2}$. (3)

QUESTION 5

The following geometric sequence is given: 10 ; 5 ; 2,5 ; 1,25 ; ...

(a) Calculate the value of the 5^{th} term, T_5, of this sequence. (2)

(b) Determine the n^{th} term, T_n, in terms of n. (2)

(c) Explain why the infinite series 10 + 5 + 2,5 + 1,25 + ... converges. (2)

(d) Determine $S_\infty - S_n$ in the form ab^n, where S_n is the sum of the first n terms of the sequence. (4)

QUESTION 6

Consider the series: $S_n = -3+5+13+21+...$ to *n terms*.

(a) Determine the general term of the series in the form $T_k = bk + c$. (2)

(b) Write S_n in sigma notation. (2)

(c) Show that $S_n = 4n^2 - 7n$. (3)

(d) Another sequence is defined as:

$$\begin{aligned} Q_1 &= -6 \\ Q_2 &= -6-3 \\ Q_3 &= -6-3+5 \\ Q_4 &= -6-3+5+13 \\ Q_5 &= -6-3+5+13+21 \end{aligned}$$

(1) Write down a numerical expression for Q_6. (2)

(2) Calculate the <u>value</u> of Q_{129}. (3)

QUESTION 7

Given $f(x) = 3(x-1)^2 + 5$ and $g(x) = 3$

(a) Is it possible for $f(x) = g(x)$? Given a reason for your answer. (2)

(b) Determine the value(s) of k for which $f(x) = g(x) + k$ has two unequal real roots. (2)

QUESTION 8

Given arithmetic series $18 + 24 + 30 + \cdots + 300$

(a) Determine the number of terms in the series. (3)

(b) Calculate the sum of this series . (2)

(c) Calculate the sum of all the whole numbers up to and including 300 that are not divisible by six. (4)

QUESTION 9

The sum of n terms is given by $S_n = \frac{n}{2}(1+n)$, find T_5. (3)

SECTION B: NOTES ON CONTENT

Arithmetic Sequences and Series

An arithmetic sequence or series is the linear number pattern discussed in Grade 10. We have a formula to help us determine any specific term of an arithmetic sequence. We also have formulae to determine the sum of a specific number of terms of an arithmetic series.

The formulae are as follows:

$T_n = a + (n-1)d$ where $a =$ first term and $d =$ constant difference

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ where $a =$ first term and $d =$ constant difference

$S_n = \frac{n}{2}\left[a + l\right]$ where l is the last term

Geometric Sequences and Series

We have a formula to help us determine any specific term of a geometric sequence. We also have formulae to determine the sum of a specific number of terms of a geometric series.

The formulae are as follows:

$T_n = ar^{n-1}$

$S_n = \frac{a(r^n - 1)}{r - 1}$ where $r \neq 1$

Convergent geometric series

Consider the following geometric series:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+................$$

We can work out the sum of progressive terms as follows:

$S_1=\frac{1}{2}=0,5$ (Start by adding in the first term)

$S_2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}=0,75$ (Then add the first two terms)

$S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}=0,875$ (Then add the first three terms)

$S_4=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}=0,9375$ (Then add the first four terms)

If we continue adding progressive terms, it is clear that the decimal obtained is getting closer and closer to 1. The series is said to converge to 1. The number to which the series converges is called the <u>sum to infinity</u> of the series.
There is a useful formula to help us calculate the sum to infinity of a convergent geometric series.

The formula is $S_\infty=\frac{a}{1-r}$

If we consider the previous series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+................$

It is clear that $a=\frac{1}{2}$ and $r=\frac{1}{2}$

$$S_\infty=\frac{a}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

A geometric series will converge only if the constant ratio is a number between negative one and positive one.
In other words, the sum to infinity for a given geometric series will exist only if $-1<r<1$. If the constant ratio lies outside this interval, then the series will not converge.

SECTION C: ACTIVITIES

QUESTION 1

The 19th term of an arithmetic sequence is 11, while the 31st term is 5.

(a) Determine the first three terms of the sequence. (5)

(b) Which term of the sequence is equal to -29? (3)

QUESTION 2

Given: $\frac{1}{181}+\frac{2}{181}+\frac{3}{181}+\frac{4}{181}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{180}{181}$

(a) Calculate the sum of the given series. (4)

(b) Hence calculate the sum of the following series:

$$\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\ldots\ldots+\left(\frac{1}{181}+\frac{2}{181}+\ldots\ldots+\frac{180}{181}\right)$$ (4)

QUESTION 3 (DoE Feb 2009 Paper 1)

The following is an arithmetic sequence: $1-p\ ;\ 2p-3\ ;\ p+5\ ;\ \ldots..$

(a) Calculate the value of *p*. (3)

(b) Write down the value of:

(1) The first term of the sequence (1)

(2) The common difference (2)

(c) Explain why none of the numbers in this arithmetic sequence are perfect squares. (2)

QUESTION 4

In a geometric sequence in which all terms are positive, the sixth term is $\sqrt{3}$ and the eighth term is $\sqrt{27}$. Determine the first term and constant ratio. (7)

QUESTION 5

(a) Determine n if: $\sum_{r=1}^{n}(6r-1)=456$ (7)

(b) Prove that: $\sum_{k=3}^{n}(2k-1)n=n^3-4n$. (6)

(c) Write the following series in sigma notation: $2+5+8+11+14+17$ (4)

QUESTION 6

Consider the series $\sum_{n=1}^{\infty} 2(\frac{1}{2}x)^n$

(a) For which values of x will the series converge? (3)

(b) If $x=\frac{1}{2}$, calculate the sum to infinity of this series. (3)

QUESTION 7 (DoE Feb 2009 Paper 1)

A sequence of squares, each having side 1, is drawn as shown below. The first square is shaded, and the length of the side of each shaded square is half the length of the side of the shaded square in the previous diagram.

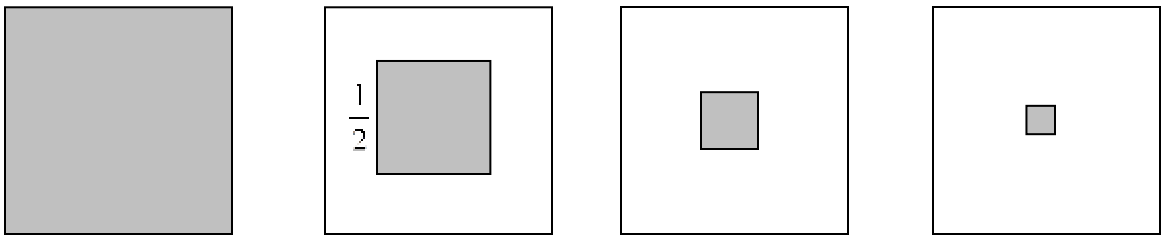

DIAGRAM 1 **DIAGRAM 2** **DIAGRAM 3** **DIAGRAM 4**

(a) Determine the area of the unshaded region in DIAGRAM 3. (2)

(b) What is the sum of the areas of the unshaded regions on the first seven squares? (5)

QUESTION 8

A plant grows 1,5 m in 1st year. It's growth each year thereafter is $\frac{2}{3}$ of its growth in the previous year. What is the greatest height it can reach? (3)

SECTION D: SOLUTIONS FOR SECTION A SESSION 1

1(a)(1)	$T_n = a+(n-1)d$ $\therefore T_{100} = -2+(100-1)(5) = 493$	✓ $T_n = a+(n-1)d$ ✓ $T_{100} = 493$ (2)

1(a)(2)	$S_n = \frac{n}{2}[2a+(n-1)d]$ $\therefore S_{100} = \frac{100}{2}[2(-2)+(100-1)(5)] = 24550$	✓ $S_n = \frac{n}{2}[2a+(n-1)d]$ ✓ $T_{100} = 493$ (2)
1(b)	$T_{13} = 15$ $T_7 = 51$ $\therefore a+12d = 15$ $\therefore a+6d = 51$ $\therefore a+12d = 15$ ….A $\therefore a+6d = 51$….. B $\therefore 6d = -36$ $A-B$ $\therefore d = -6$ $\therefore a+12(-6) = 15$ $\therefore a-72 = 15$ $\therefore a = 87$ $T_n = -21$ $\therefore a+(n-1)d = -21$ $\therefore 87+(n-1)(-6) = -21$ $\therefore 87-6n+6 = -21$ $\therefore n = 19$ $\therefore T_{19} = -21$	✓ $a+12d = 15$ ✓ $a+6d = 51$ ✓ $d = -6$ ✓ $a = 87$ ✓ $87+(n-1)(-6) = -21$ ✓ $n = 19$ (6)
2(a)	$T_6 = 243$ and $T_3 = 72$ $a.r^5 = 243$ …A $a.r^2 = 72$ …B $a.r^5 = 243$ …. A $a.r^2 = 72$ ……B $\therefore r^3 = \frac{27}{8}$ …. $A \div B$ $\therefore r = \frac{3}{2}$	✓ $a.r^5 = 243$ ✓ $a.r^2 = 72$ ✓ $r^3 = \frac{27}{8}$ ✓ $r = \frac{3}{2}$ (4)
2(b)	$\therefore a\left(\frac{3}{2}\right)^5 = 243$ $\therefore a = 32$ $\therefore S_{10} = \dfrac{32\left(\left(\frac{3}{2}\right)^{10}-1\right)}{\frac{3}{2}-1} = 3626,5625$	✓ $a\left(\frac{3}{2}\right)^5 = 243$ ✓ $a = 32$ ✓ $S_{10} = \dfrac{32\left(\left(\frac{3}{2}\right)^{10}-1\right)}{\frac{3}{2}-1}$ ✓ answer (4)
3(a)	$\frac{1}{16}$; 13	✓✓ answers (2)
3(b)	$S_{50} =$ 25 terms of 1st sequence + 25 terms of 2nd sequence	✓ separating into an arithmetic and geometric series

	$S_{50} = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{...to 25 terms}\right)+\left(4+7+10+13+\text{...to 25 terms}\right)$ $S_{50} = \dfrac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{25}-1\right)}{\frac{1}{2}-1}+\frac{25}{2}\left[2(4)+24(3)\right]$ $S_{50} = 0,999999...+1000$ $S_{50} = 1001,00$	✓✓ $\dfrac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{25}-1\right)}{\frac{1}{2}-1}$ ✓ correct formulae ✓✓ $\frac{25}{2}\left[2(4)+24(3)\right]$ ✓ answer (7)

4(a)	$T_n = (8x^3)\left(\frac{1}{2}x\right)^{n-1}$	✓ correct formula ✓ $a = 8x^3$ ✓ $r = \frac{1}{2}x$ (3)
4(b)	$-1 < \frac{x}{2} < 1$ $= -2 < x < 2$	✓ $-1 < \frac{x}{2} < 1$ ✓ $-2 < x < 2$ (2)
4(c)	$S_\infty = \frac{a}{1-r}$ $\therefore S_\infty = \dfrac{8x^2}{1-\frac{x}{2}}$ $\therefore S_\infty = \dfrac{8\left(\frac{3}{2}\right)^2}{1-\frac{1}{2}\left(\frac{3}{2}\right)}$ $\therefore S_\infty = 72$	✓ correct formula ✓ substitution ✓ answer (3)
5(a)	$r = \frac{T_2}{T_1}$ $= \frac{5}{10}$ $= \frac{1}{2}$ $T_5 = 1{,}25\left(\frac{1}{2}\right)$ $= \frac{5}{8}$ or $0{,}625$ OR/OF $T_5 = 10\left(\frac{1}{2}\right)^4$ $= \frac{5}{8}$ or $0{,}625$	✓ r = $\frac{1}{2}$ ✓answer (2)

5(b)	$T_n = 10\left(\frac{1}{2}\right)^{n-1}$	✓substitutes $a = 10$ into GP formula ✓substitutes $r = \frac{1}{2}$ into GP formula (2)
5(c)	r = ½ $-1 < r < 1$ Therefore the sequence converges/ Die ry konvergeer	✓$-1 < r < 1$ ✓ show that r = ½ is $-1 < r < 1$ (2)
5(d)	$S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r}$ $= \frac{10}{1-\frac{1}{2}} - \frac{10\left(1-\frac{1}{2}^n\right)}{1-\frac{1}{2}}$ $= 20 - 20\left(1-\frac{1}{2}^n\right)$ $= 20 - 20 + 20\times\left(\frac{1}{2}\right)^n$ $= 20\times\left(\frac{1}{2}\right)^n$ OR/OF $S_\infty - S_n = T_{n+1} + T_{n+2} + T_{n+3} + \ldots$ $= 10\left(\frac{1}{2}\right)^n\left[1+\frac{1}{2}+\frac{1}{4}+\ldots\right]$ $= 10\left(\frac{1}{2}\right)^n\left[\frac{1}{1-\frac{1}{2}}\right]$ $= 20\times\left(\frac{1}{2}\right)^n$ OR/OF	✓ $\frac{10}{1-\frac{1}{2}}$ ✓ $\frac{10\left(1-\frac{1}{2}^n\right)}{1-\frac{1}{2}}$ ✓ $-20+20\times\left(\frac{1}{2}\right)^n$ ✓answer (4) ✓constructing the series ✓ $10\left(\frac{1}{2}\right)^n\left[1+\frac{1}{2}+\frac{1}{4}+\ldots\right]$ ✓ $\frac{1}{1-\frac{1}{2}}$ ✓answer (4) ✓ $\frac{a-a+ar^n}{1-r}$ ✓ $\frac{ar^n}{1-r}$

	$S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r}$ $= \frac{a-a+ar^n}{1-r}$ $= \frac{ar^n}{1-r}$ $= \frac{10\left(\frac{1}{2}\right)^n}{\frac{1}{2}}$ $= 20\times\left(\frac{1}{2}\right)^n$	✓ $\frac{10\left(\frac{1}{2}\right)^n}{\frac{1}{2}}$ ✓answer (4)
6(a)	$T_k = a + (k-1)d$ $= -3 + (k-1)(8)$ $= -3 + 8k - 8$ $= 8k - 11$	✓d value ✓answer (2)
6(b)	$\sum_{k=1}^{n}(8k-11)$ **OR/OF** $\sum_{k=0}^{n-1}(8(k+1)-11) = \sum_{k=0}^{n-1}(8k-3)$	✓for general term ✓lower and upper values in sigma notation (2)
6(c)	$S_n = \frac{n}{2}[2a + (n-1)d]$ $= \frac{n}{2}[2(-3) + (n-1)(8)]$ $= \frac{n}{2}[-6 + 8n - 8]$ $= \frac{n}{2}[8n - 14]$ $= n(4n-7)$ $= 4n^2 - 7n$ OR/OF	✓formula ✓substitution ✓ $\frac{n}{2}[8n-14]$ (3)

	$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ $= \frac{n}{2}\left[2(-3) + (n-1)(8)\right]$ $= \frac{n}{2}\left[-6 + 8n - 8\right]$ $= \frac{n}{2}\left[8n - 14\right]$ $= 4n^2 - 7n$ OR/OF $S_n = \frac{n}{2}\left[a + l\right]$ $= \frac{n}{2}\left[-3 + 8n - 11\right]$ $= \frac{n}{2}\left[8n - 14\right]$ $= 4n^2 - 7n$ OR/OF S_1 S_2 S_3 S_4 -3 2 15 36 5 13 21 8 8 $S_n = an^2 + bn + c$ $a = 4$ $S_1 = 4 + b + c = -3$ $b + c = -7$ ....(1) $S_2 = 16 + 2b + c = 2$ $2b + c = -14$.............(2) $b = -7$...............(2)−(1) $c = 0$ Hence $S_n = 4n^2 - 7n$	✓formula ✓substitution ✓ $\frac{n}{2}\left[8n - 14\right]$ (3) ✓formula ✓substitution ✓ $\frac{n}{2}\left[8n - 14\right]$ (3) ✓calculates S1, S2, S3 and S4, ✓ a = 4 ✓solves simultaneously for b and c. (3)
6(d) 1	$Q_6 = -6 - 3 + 5 + 13 + 21 + 29$	✓✓ answer (2)

<table>
<tr>
<td>6(d) 2</td>
<td>
$Q_{129} = -6 + S_{128}$
$= -6 + 4(128)^2 - 7(128)$
$= 64634$
OR/OF
Q_1 Q_2 Q_3 Q_4
-6 -9 -4 9
-3 5 13
8 8 8
$Q_n = an^2 + bn + c$
$a = 4$
$Q_1 = 4 + b + c = -6$ $b + c = -10$ ....(1)
$Q_2 = 16 + 2b + c = -9$ $2b + c = -25$..............(2)
$b = -15$...............(2)$-$(1)
$c = 5$
Hence $Q_n = 4n^2 - 15n + 5$
$Q_{129} = 4(129)^2 - 15(129) + 5$
$= 64\ 634$
</td>
<td>
✓✓ $-6 + 4(128)^2 - 7(128)$
✓answer
(3)

✓a = 4

✓ $Q_n = 4n^2 - 15n + 5$

✓answer
(3)
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<td>7(a)</td>
<td>
No, there will be no intersection between the graphs.
Minimum value of $3(x-1)^2 + 5$ *is* 5

Nee, daar saal geen snyding tussen die grafieke wees nie, Min waarde van
$3(x-1)^2 + 5$ *is* 5

OR/ OF
$$3(x-1)^2 + 5 = 3$$
$$3(x-1)^2 = -2$$
$$(x-1)^2 = -\frac{2}{3}$$
No, there will be no intersection between the graphs.

Nee, daar sal geen snyding tussen die grafieke wees nie.
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✓ answer
✓ reason

✓ answer
✓ reason

[2]
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7 (b)	$3(x-1)^2+5=3+k$ $3(x-1)^2=k-2$ $k-2>0$ $k>2$ For all real values of x / $vir\ alle\ reeele\ waardes\ van\ x$	✓ answer ✓ reason [2]

8(a)	$T_n=a+(n-1)d$ $300=18+(n-1)6$ $300=18+6n-6$ $6n=288$ $n=48$	✓ $a=18$ & d=6 ✓ Answer ✓ 300 [3]
8(b)	$S_n=\frac{n}{2}[2a+(n-1)d]$ $S_{48}=\frac{48}{2}[2(18)+(47)6]$ $S_{48}=7632$	✓ $subst$ ✓ answer [2]
8(c)	Sum of all numbers from 1 to 300 $Som\ van\ allegetalle\ van\ 1\ tot\ 300$ $S_{300}=\frac{300}{2}[2(1)+(299)(1)]$ $S_{300}=\frac{300(301)}{2}$ $S_{300}=45150$ Sum of numbers not divisible by 6 / $Som\ van\ die\ getalle\ wat\ nie\ deelbaar\ deur\ 6\ is\ nie.$ $=45150-(7632+6+12)$ $=37500)$	✓ $subst$ ✓ answer ✓ $(7632+6+12)$ ✓ answer [4]

9	$S_5=\frac{5}{2}(1+5)=15$ $S_4=\frac{4}{2}(1+4)=10$ $\therefore T_5=15-10=5$	✓ 15 ✓ 10 ✓ 5 [3]

SESSION NO: 2

TOPIC: FUNCTIONS AND INVERSE FUNCTIONS

Teacher Note: Changing from exponential to logarithmic form in real world problems is the most important concept in this section. This concept is particularly useful in Financial Maths when required to solve for *n*. Make sure that learners understand this concept thoroughly.

LESSON OVERVIEW

1.	**Introduction session**	**5 minutes**
2.	**Typical exam questions and solutions:**	**Remaining time**

SECTION A: TYPICAL EXAM QUESTIONS

QUESTION 1

Consider the functions: $f(x) = 2x^2$ and $g(x) = \left(\frac{1}{2}\right)^x$

(a) Restrict the domain of f in one specific way so that the inverse of f will also be a function. (1)

(b) Hence draw the graph of your new function *f* and its inverse function f^{-1} on the same set of axes. (2)

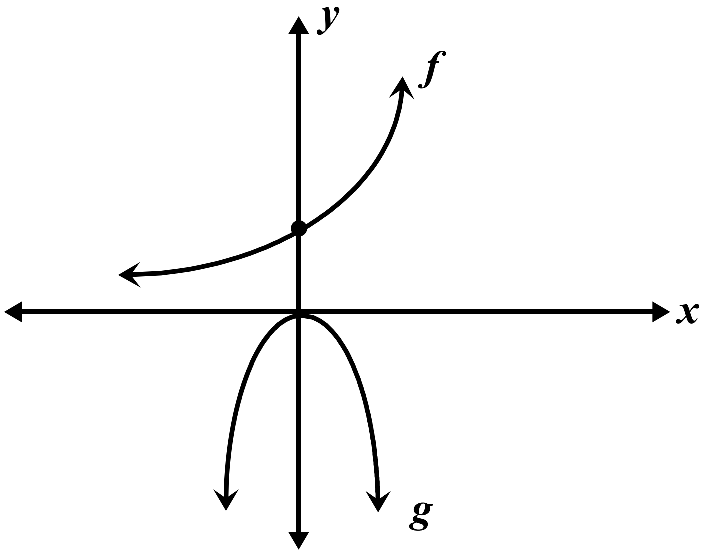
(c) Write the inverse of *g* in the form $g^{-1}(x) = ...$ (2)

(d) Sketch the graph of g^{-1}. (2)

(e) Determine graphically the values of *x* for which $\log_{\frac{1}{2}} x < 0$ (1)

QUESTION 2

Sketched below are the graphs of $f(x) = 3^x$ and $g(x) = -x^2$



(a) Write down the equation of the inverse of the graph of $f(x) = 3^x$ in the form $f^{-1}(x) = ...$ (2)

(b) On a set of axes, draw the graph of the inverse of $f(x) = 3^x$ (2)

(c) Write down the domain of the graph of $f^{-1}(x)$ (1)

(d) Explain why the inverse of the graph of $g(x) = -x^2$ is not a function. (1)

(e) Consider the graph of $g(x) = -x^2$

(1) Write down a possible restriction for the domain of $g(x) = -x^2$ so that the inverse of the graph of *g* will now be a function. (1)

(2) Hence draw the graph of the inverse function in (1). (2)

QUESTION 3

Given: $g(x) = \left(\frac{1}{2}\right)^x$

(a) Write the inverse of *g* in the form $g^{-1}(x) = ...$ (2)

(b) Sketch the graph of g^{-1} (2)

(c) Determine graphically the values of *x* for which $\log_{\frac{1}{2}} x < 0$ (1)

QUESTION 4

Given: $f(x) = 2^{x+1} - 8$

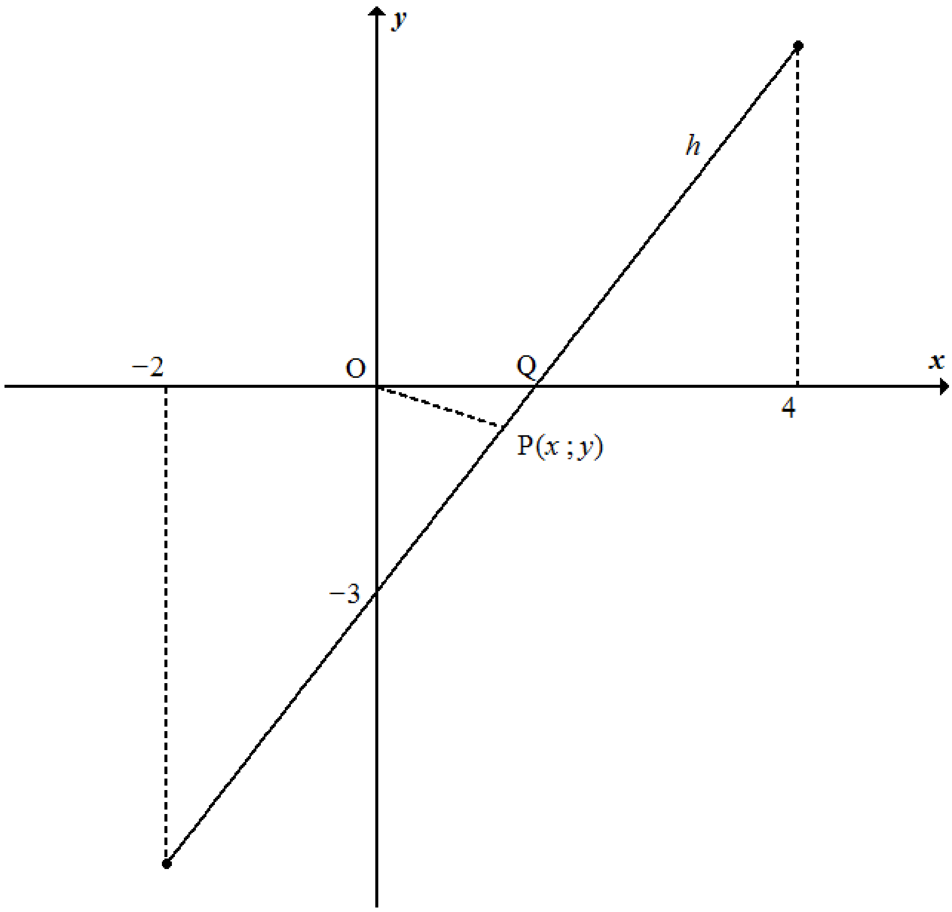
(a) Write down the equation of the asymptote of *f*. (1)

(b) Sketch the graph of *f*. Clearly indicate ALL intercepts with the axes as well as the asymptote. (4)

(c) The graph of *g* is obtained by reflecting the graph of *f* in the *y*-axis. Write down the equation of *g*. (1)

QUESTION 5

Given: $h(x) = 2x - 3$ for $-2 \leq x \leq 4$. The *x*-intercept of *h* is Q.



(a) Determine the coordinates of Q. (2)

(b) Write down the domain of h^{-1}. (3)

(c) Sketch the graph of h^{-1} in your ANSWER BOOK, clearly indicating the y-intercept and the end points. (3)

(d) For which value(s) of x will $h(x) = h^{-1}(x)$? (3)

(e) P(x ; y) is the point on the graph of h that is closest to the origin. Calculate the distance OP. (5)

QUESTION 6

The function defined as $y = \dfrac{a}{x+p} + q$ has the following properties:

- The domain is $x \in R,\ x \neq -2.$
- $y = x + 6$ is an axis of symmetry.
- The function is increasing for all $x \in R,\ x \neq -2.$

Draw a neat sketch graph of this function. Your sketch must include the asymptotes, if any. (4)

SECTION B: NOTES ON CONTENT

It is important for learners to understand what the concept of a function is. When the graphs of the inverses of many-to-one functions are drawn, the graph is no longer a function. In order to ensure that the graph of the inverse of a many-to-one function is also a function, it is necessary to first restrict the domain of the original function. However, what is important to note is that the **range** of the restricted function must remain the same as the range of the original function. Remind your learners that the graph of the inverse of a function is a reflection about the line $y = x$.

Emphasize the fact that the inverse of an exponential function is also a function and that the graph of this inverse function is called a logarithmic function. Make sure your learners know how to determine the domain and range of functions and their inverses.

Logarithms need not be studied in too much detail. Teach learners the definition of a logarithm and ensure that they know how to move from log-form to exponential form and exponential form to log-form. Make them aware of the four laws as well as the deductions and let them practice a few basic exercises, but don't go into too much detail. What is important is the use of the definition and the laws in real-world situations including exponential growth, decay and financial calculations.

If a number is written in exponential form, then the exponent is called the logarithm of the number. For example, the number 64 can be written in exponential form as $64 = 2^6$. Clearly, the exponent in this example is 6 and the base is 2. We can then say that the logarithm of 64 to base 2 is 6. This can be written as $\log_2 64 = 6$.
The base 2 is written as a sub-script between the "log" and the number 8.
In general, we can rewrite $\text{number} = \text{base}^{\text{exponent}}$ in logarithmic form as follows:

$$\text{number} = \text{base}^{\text{exponent}}$$

$$\therefore \log_{\text{base}}(\text{number}) = \text{exponent}$$

SECTION C: ACTIVITIES

QUESTION 1

A colony of an endangered species originally numbering 1000 was predicted to have a population N after *t* years given by the equation $\mathrm{N} = 1000(0{,}9)^t$.

(a) Estimate the population after 1 year. (2)

(b) Estimate the population after 2 years. (2)

(c) After how many years will the population decrease to 200? (5)

QUESTION 2

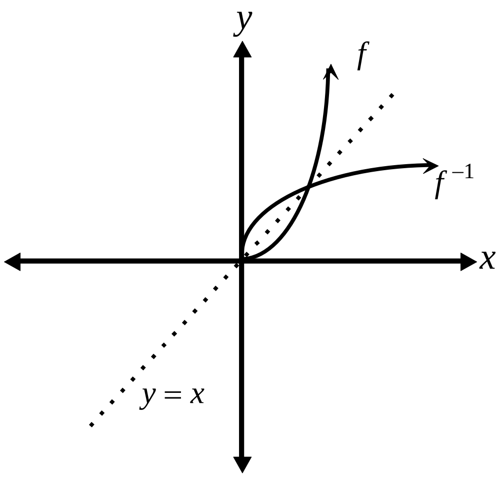
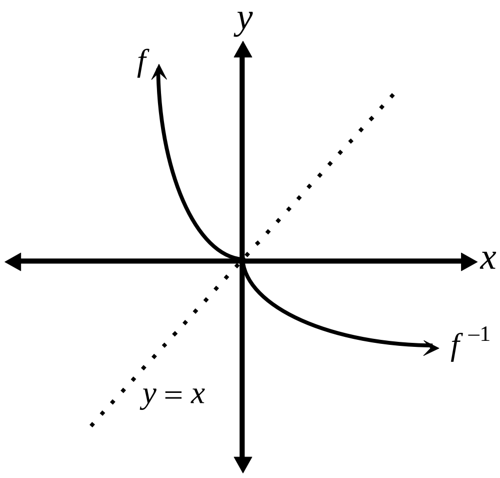
The graph of $f : x \to \log_a x$ passes through the point (16 ; 2).

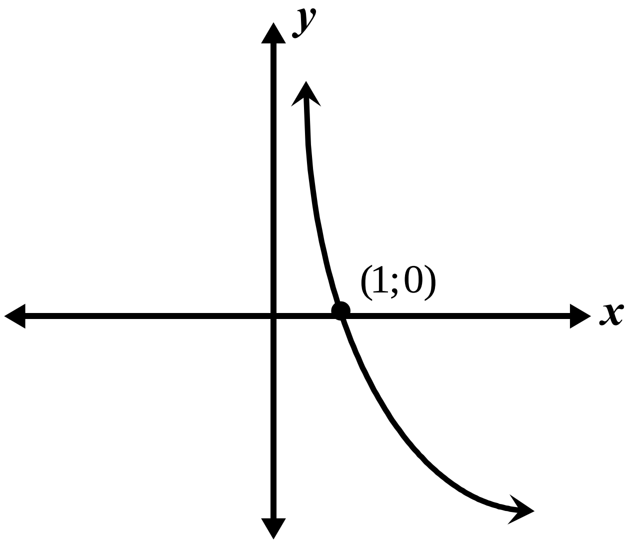
(a) Calculate the value of a. (3)

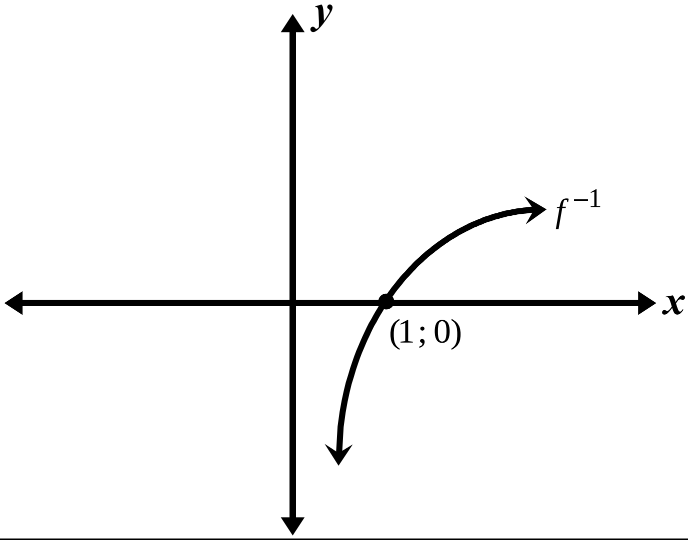
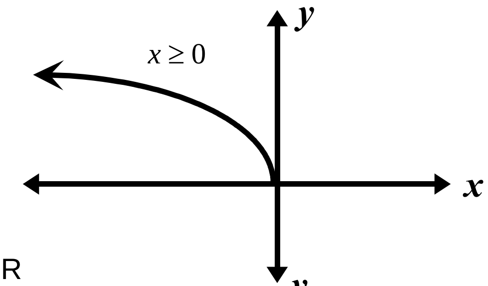
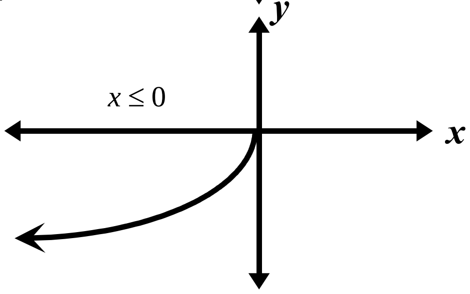
(b) Write down the equation of the inverse in the form $f^{-1}(x) = ...$ (2)

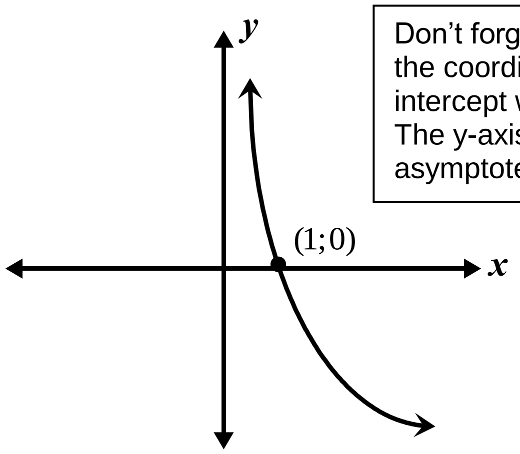
SECTION D: SOLUTIONS FOR SECTION A SESSION 2

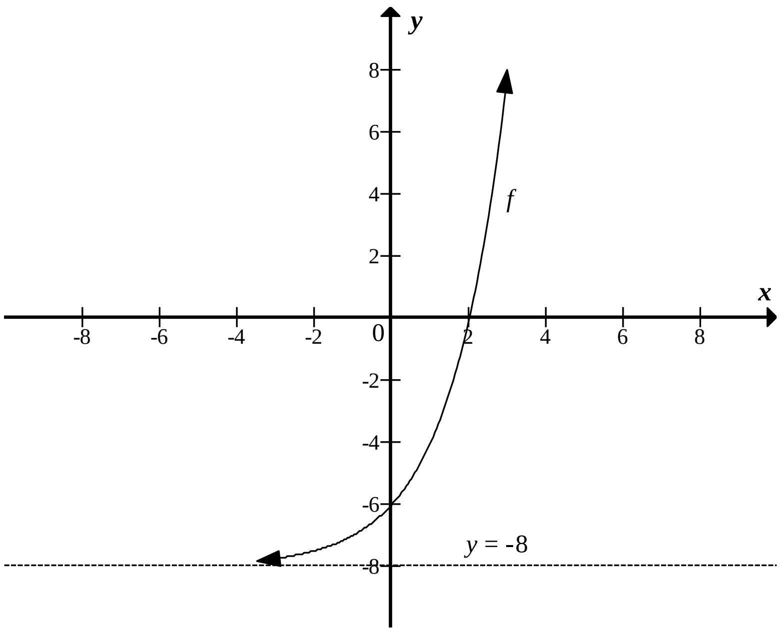
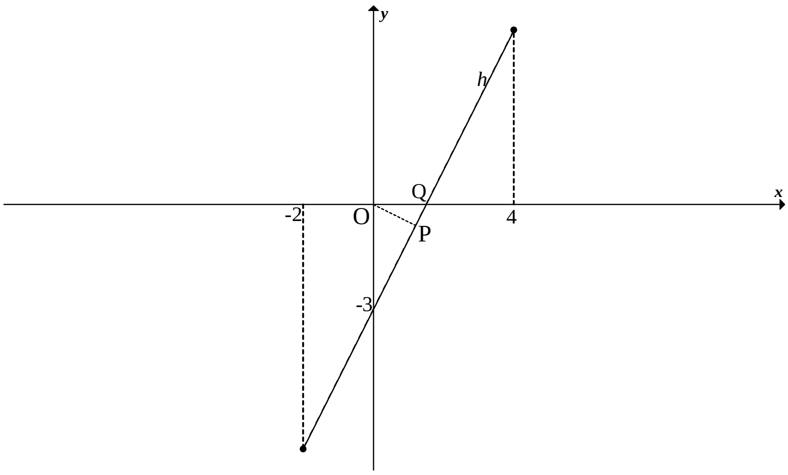
1(a)	$f(x)=2x^2$ where $x \geq 0$ OR $f(x)=2x^2$ where $x \leq 0$	✓ $x \geq 0$ OR $x \leq 0$ (1)

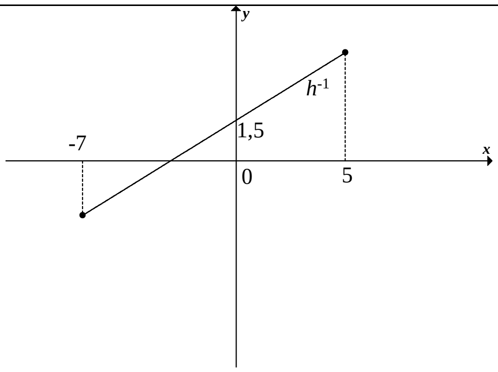
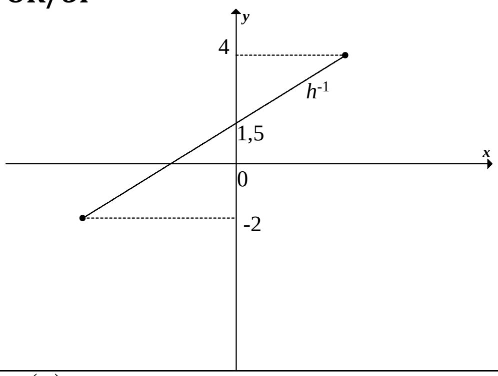
1(b)	 OR  	✓ f ✓ f^{-1} (2)

1(c)	$y=\left(\frac{1}{2}\right)^{x}$ $x=\left(\frac{1}{2}\right)^{y}$ $\therefore \log_{\frac{1}{2}} x = y$ $\therefore g^{-1}(x)=\log_{\frac{1}{2}} x$	✓ $x=\left(\frac{1}{2}\right)^{y}$ ✓ $g^{-1}(x)=\log_{\frac{1}{2}} x$ (2)
1(d)		✓ shape ✓ $(1;0)$ (2)
1(e)	$\log_{\frac{1}{2}} x<0$ for $x>1$	✓ $x>1$ (1)

2(a)	$y = 3^x$ $\therefore x = 3^y$ $\therefore \log_3 x = y$ $\therefore f^{-1}(x) = \log_3 x$	✓ $x = 3^y$ ✓ $f^{-1}(x) = \log_3 x$ (2)
2(b)		✓ shape ✓ (1 ; 0) (2)
2(c)	Domain: $x \in (0\,;\infty)$	✓ $x \in (0\,;\infty)$ (1)
2(d)	The inverse is a one-to-many relation, which is not a function.	✓ one-to-many (1)
2(e)(1)	$x \geq 0$ OR $x \leq 0$	✓ answer (1)
2(e)(2)	 OR 	✓ Shape ✓ origin (2)

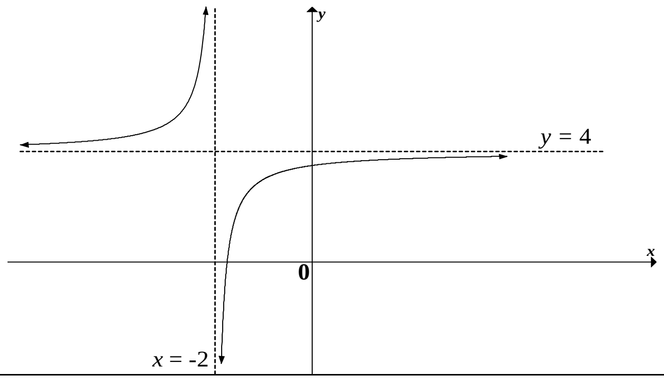
3(a)	$y=\left(\frac{1}{2}\right)^x$ $x=\left(\frac{1}{2}\right)^y$ $\therefore \log_{\frac{1}{2}} x = y$ $\therefore g^{-1}(x)=\log_{\frac{1}{2}} x$ Remember that the inverse of a graph is determined by interchanging x and y in the equation of the original graph.	✓ $x=\left(\frac{1}{2}\right)^y$ ✓ $g^{-1}(x)=\log_{\frac{1}{2}} x$ (2)
3(b)	 Don't forget to indicate the coordinates of the intercept with the axes. The y-axis is the asymptote of the graph.	✓ shape ✓ $(1;0)$ (2)
3(c)	$\log_{\frac{1}{2}} x<0$ for $x>1$	✓ $x>1$ (1)

Given: $f(x)=2^{x+1}-8$		
4(a)	$y=-8$	✓ $y=-8$ (1)
4(b)		✓x-intercept ✓y-intercept ✓shape ✓asymptote (only if the graph does not cut the asymptote) (4)
4(c)	$g(x)=2^{-x+1}-8$ **OR/*OF*** $g(x)=\left(\frac{1}{2}\right)^{x-1}-8$	✓answer (1) ✓answer (1)
QUESTION 5		
Given $h(x)=2x-3$ for $-2\leq x\leq 4$. 		

5(a)	For x-intercepts, $y = 0$ $2x - 3 = 0$ $x = 1{,}5$ $Q(1{,}5\,;\,0)$	✓ $x = 1{,}5$ ✓ $y = 0$ (2)
5(b)	$h:$ $x = -2: \quad y = 2(-2) - 3 = -7$ $x = 4: \quad y = 2(4) - 3 = 5$ Domain of $h^{-1}: -7 \leq x \leq 5$ **OR/*OF*** $[-7;\,5]$	✓ $h(-2) = -7$ ✓ $h(4) = 5$ ✓ $-7 \leq x \leq 5$ (3)
5(c)	 **OR/*OF*** 	✓ y-intercept on a straight line ✓ line segment ✓ accurate endpoints (x or y or both) (3)
5(d)	$h(x) = 2x - 3$ For the inverse of h, $x = 2y - 3$ $y = \dfrac{x+3}{2}$ $h(x) = h^{-1}(x)$ $2x - 3 = \dfrac{x+3}{2}$ $4x - 6 = x + 3$ $3x = 9$ $x = 3$	✓ $y = \dfrac{x+3}{2}$ ✓ $2x - 3 = \dfrac{x+3}{2}$ ✓ $x = 3$ (3)

	OR/*OF*	
	$h(x)=2x-3$ h and h^{-1} intersect when $y=x$ $h(x)=x$ $2x-3=x$ $x=3$ **OR/*OF*** $h(x)=2x-3$ For the inverse of h, $x=2y-3$ $y=\frac{x+3}{2}$ $h^{-1}(x)=x$ $\frac{x+3}{2}=x$ $x+3=2x$ $x=3$	✓ $h(x)=x$ ✓ $2x-3=x$ ✓ $x=3$ (3) ✓ $y=\frac{x+3}{2}$ ✓ $\frac{x+3}{2}=x$ ✓ $x=3$ (3)
5(e)	$\text{OP}^2=(x-0)^2+(y-0)^2$ $=x^2+(2x-3)^2$ $=x^2+4x^2-12x+9$ $=5x^2-12x+9$ For OP to be at its minimum, OP^2 has to be a minimum *Vir OP om minimum te wees, moet OP^2 'n minimum wees* $x=-\frac{b}{2a}$ $=-\frac{-12}{2(5)}$ $\therefore x=\frac{6}{5}$ Minimum length of OP $=\sqrt{5\left(\frac{6}{5}\right)^2-12\left(\frac{6}{5}\right)+9}=\sqrt{\frac{9}{5}}$ or $\frac{3}{\sqrt{5}}$ or 1,34 units **OR/*OF***	✓ $\text{OP}^2=x^2+y^2$ ✓substitute $y=2x-3$ ✓ $5x^2-12x+9$ ✓x-value ✓answer (5)

	$m_h = 2$ (given) $m_{\text{OP}} = \frac{-1}{2}$ $\therefore$ OP has equation $y = \frac{-1}{2}x$ $\frac{-1}{2}x = 2x - 3$ $-x = 4x - 6$ $5x = 6$ $x_P = 1{,}2$ $y_P = -\frac{1}{2}(1{,}2) = -0{,}6$ $\text{OP} = \sqrt{(1{,}2-0)^2 + (-0{,}6-0)^2}$ $= 1{,}34$ or $\sqrt{1{,}8}$ units **OR/*OF*** $O(0;0)$ $P(x;\ 2x-3)$ $Q\left(\frac{3}{2};0\right)$ $OP^2 + PQ^2 = OQ^2$ (pythag) $(x-0)^2 + (2x-3-0)^2 + \left(x - \frac{3}{2}\right)^2 + (2x-3-0)^2 = \left(\frac{3}{2}\right)^2$ $x^2 + 4x^2 - 12x + 9 + x^2 - 3x + \frac{9}{4} + 4x^2 - 12x + 9 = \frac{9}{4}$ $10x^2 - 27x + 18 = 0$ $(5x-6)(2x-3) = 0$ $x = \frac{6}{5}$ *or* $\frac{3}{2}$ Hence, $x = \frac{6}{5}$ at P	✓ $m_{\text{OP}} = \frac{-1}{2}$ ✓equation of OP ✓ $\frac{-1}{2}x = 2x - 3$ ✓x-value ✓answer (5) ✓ $\text{OP}^2 = x^2 + y^2$ ✓substitute $y = 2x - 3$ ✓ $10x^2 - 27x + 18$

	$OP^2 = x^2 + (2x-3)^2$ $= \left(\frac{6}{5}\right)^2 + \left(2\left(\frac{6}{5}\right) - 3\right)^2$ $= \frac{36}{25} + \frac{9}{25}$ $= \frac{9}{5}$ $OP = 1{,}34$ **OR/*OF*** $\tan \hat{Q} = 2$ $\hat{Q} = 63{,}43^\circ$ $\sin 63{,}43^\circ = \frac{OP}{1{,}5}$ $OP = 1{,}34$	✓x-value ✓answer (5)
QUESTION 6		
6	If C$(x; y)$is the centre of the hyperbola/*As C$(x; y)$die middelpunt is van die hiperbool* $y = x + 6$ and $x = -2$ $\therefore y = -2 + 6 = 4$ 	✓✓ asymptote $y = 4$ ✓asymptote $x = -2$ ✓shape (increasing hyperbolic function) (4)