Module 4: Financial Mathematics

CAPS extraction indicating progression from Grades 10-12

Grade 10	Grade 11	Grade 12
Use simple and compound growth formulae $A = P(1+in)$ and $A = P(1+i)^n$ to solve problems (including interest, hire purchase, inflation, population growth and other real life problems).	Use simple and compound growth formulae $A = P(1+in)$ and $A = P(1+i)^n$ to solve problems (including straight line depreciation and depreciation on a reducing balance) The effect of different periods of compounding growth and decay (including effective and nominal interest rates).	Calculate the value of n in the formula $A = P(1+i)^n$ and $A = P(1-i)^n$ Apply knowledge of geometric series to solve annuity and bond repayment problems. Critically analyse different loan options.

Introduction

The study of Financial Mathematics is centred on the concepts of simple and compound growth. The learner must be made to understand the difference in the two concepts at Grade 10 level. This may then be successfully built upon in Grade 11, eventually culminating in the concepts of Present and Future Value Annuities in Grade 12.

One of the most common misconceptions found in the Grade 12 examinations is the lack of understanding that learners have from the previous grades (Grades 10 and 11) and the lack of ability to manipulate the formulae. In addition to this, many learners do not know when to use which formulae, or which value should be allocated to which variable. Mathematics is becoming a subject of rote learning that is dominated by past year papers and memorandums which deviate the learner away from understanding the basic concepts, which make application thereof simple.

Let us begin by finding ways in which we can effectively communicate to learners the concept of simple and compound growth.

Simple and Compound Growth

- What is our understanding of simple and compound growth?
- How do we, as educators, effectively transfer our understanding of these concepts to our learners?
- What do the learners need to know before we can begin to explain the difference in simple and compound growth?

A star educator always takes into account the dynamics of his/her classroom

The first aspect that learners need is to understand the terminology that is going to be used.

Activity 1: Terminology for Financial Maths

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to write down definitions/explanations that you will use in your classroom to explain to your learners the meaning of the following terms:

- Interest
- Principal amount
- Accrued amount
- Interest rate
- Term of investment
- Per annum

3. Every group will have an opportunity to provide feedback.

Notes:

Now that we have a clear understanding of the terms that we are going to use, let us try and understand the difference between Simple and Compound growth.

We will make use of an example to illustrate the difference between these two concepts.

Worked Example 1: Simple and Compound Growth (20 min)

The facilitator will now provide you with a suitable example to help illustrate the difference between simple and compound growth.

Example 1:

Cindy wants to invest $R500$ in a savings plan for four years. She will receive 10% interest per annum on her savings. Should Cindy invest her money in a simple or compound growth plan?

Solution:

Simple growth plan:

Interest is calculated at the start of the investment based on the money she is investing and WILL REMAIN THE SAME every year of her investment.

$$\begin{aligned} Interest &= 500 \times \left(\frac{10}{100}\right) \\ &= R50 \end{aligned}$$

This implies that every year, R50 will be added to her investment.

Year 1 : R500 + R50 = R550
Year 2 : R550 + R50 = R600
Year 3 : R600 + R50 = R650
Year 4 : R650 + R50 = R700

Notice that the interest remains the same every year.

Cindy will have an **ACCRUED AMOUNT** of R700. Her **PRINCIPAL AMOUNT** was R500.

Compound Growth Plan:

The compound growth plan has interest that is recalculated every year based on the money that is in the account. The interest WILL CHANGE every year of her investment.

Year 1:

$$Interest = 500 \times \left(\frac{10}{100}\right)$$
$$= R50$$

Therefore, at the end of the 1st year Cindy will have R500 + R50 = R550

Year 2:

$$Interest = 550 \times \left(\frac{10}{100}\right)$$
$$= R55$$

Notice that the interest is recalculated based on the amount present in the account.

Therefore, at the end of the 2nd year Cindy will have R550 + R55 = R605

Year 3:

$$Interest = 605 \times \left(\frac{10}{100}\right)$$
$$= R60.50$$

Notice that the interest is recalculated based on the amount present in the account.

Therefore, at the end of the 3rd year Cindy will have R605 + R60.50 = R665.50

Year 4:

$$Interest = 665.50 \times \left(\frac{10}{100}\right)$$
$$= R66.55$$

Notice that the interest is recalculated based on the amount present in the account.

Therefore, at the end of the 4th year Cindy will have R665.50 + R66.55 = R732.05

Cindy will have an **ACCRUED AMOUNT** of R732.05. Her **PRINCIPAL AMOUNT** was R500.

Now that we understand the difference between simple and compound growth it is evident that if we are required to perform a simple or compound growth calculation, it would be tiresome to conduct that calculation in the above manner. We will use the following formulae to help us simplify our calculations.

SIMPLE GROWTH: $A = P(1 + in)$

COMPOUND GROWTH: $A = P(1 + i)^n$

In both formulae:
A = Accrued amount
P = Principal amount
i = Interest rate
n = Number of times interest is calculated

Common Errors:

In applying these formulae, some of the most common errors found are as follows:

1. The accrued amount is the amount that will be received at the end of the investment period. This is NOT the same as the interest earned. Many times a question will ask what was the interest earned and the learner will provide the accrued amount as the answer. The accrued amount is actually the principal amount plus the interest: ($A = P + I$).

2. The interest rate is always divided by 100 in all calculations, since it is given as a percentage. We should perhaps modify the equation to be $A = P\left(1 + \left(\frac{i}{100}\right)n\right)$ and $A = P\left(1 + \frac{i}{100}\right)^n$ respectively so that the learners do not forget to divide the interest rate.

3. Learners need to understand that interest can work for and against an individual. It works to an individual's benefit when they invest a sum of money and works against them when they borrow a sum of money. Ensure that the learner understands that when money is borrowed the ACCRUED AMOUNT is the amount that has to be paid back and the PRINCIPAL AMOUNT is the initial amount that was borrowed.

4. The variable n is often mistaken for the period of the investment. This is, however, very misleading. For example, if a person invests money for two years where it is compounded monthly, the value of $n = 24$ instead of 2. Therefore, a more correct definition for n should be the number of times interest is calculated.

5. Interest may be compounded:
   Bi-annually (twice a year)
   Semi-annually (twice a year)
   Quarterly (four times a year)
   Monthly (12 times a year)
   Daily (365 times a year)
   These terms will all affect the value of n.

Worked Example 2: Simple and Compound Growth (20 min)

The facilitator will now explain this example to you.

- Remember to make notes as the facilitator is talking and ask as many questions as possible to clarify any misconceptions that may occur.

Example 2:

Thabo wanted to buy some household items that were on sale but did not have enough money. He needed R5 000 and had the following options available to him:

Option 1:

A friend agreed to lend Thabo the money at a simple interest rate of 15% per annum for a period of two years.

Option 2:

Thabo could borrow the money from a mashonisa (a loan shark) who will charge him 5% compound interest per month also for a two year period.

Thabo decided to go with option 2 as the interest rate was lower than what his friend offered him. Did Thabo make the correct decision? Use calculations to support your answer.

Solution:

<u>Option 1:</u>

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\begin{aligned}
A &= P\left(1+\left(\frac{i}{100}\right)n\right)\\
&= 5000\left(1+\left(\frac{15}{100}\right)2\right)\\
&= 5000(1+(0.15\times 2))\\
&= 5000(1.3)\\
&= R6500
\end{aligned}
$$

$$\frac{R6500}{24} = R270.83$$

If Thabo chose OPTION 1 he would pay back R270.83 per month and a total amount of R6500.

<u>Option 2:</u>

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\begin{aligned}
A &= P\left(1+\frac{i}{100}\right)^{n}\\
&= 5000\left(1+\frac{5}{100}\right)^{24}\\
&= 5000(1.05)^{24}\\
&= R16125.50
\end{aligned}
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$$\frac{R16125.50}{24} = R671.90$$

From the calculations it is evident that Thabo made the wrong choice. With option 2, he will pay a total of R9625.90 more and a monthly amount of R401.07 more than if he had chosen Option 1.

Activity 2: Calculations on Misconceptions and Errors			
Group organisation:	**Time:**	**Resources:**	**Appendix:**
Individual	30 min	• Participant hand-out • Pens/pencils • Calculators	None

This activity is to be completed individually. You are already aware of the common errors and misconceptions that we have covered. Use your calculator to answer the questions below. Note the types of misconceptions that could occur and how you would deal with these. Selected participants will share their responses with the entire group.

Answer the following questions. Show all calculations.

I. Pavan wants to invest money at an institute that offers him 3.75% interest compounded quarterly. He invests R8 000 for a period of 4 years. What is the interest that he will earn in this time period?

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Misconception: ____

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Addressing the misconception:

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II. A certain amount invested at 4.2% interest compounded semi-annually yields a return of R12 400 after five years. How much was initially invested?

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Misconception: ____

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Addressing the misconception:

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III. What is the interest rate if Nosipho invested R2 000 at simple interest for a period of three years, and received R2 800?

Misconception:

Addressing the misconception:

IV. Mr Sithole wants to buy a fridge and television amounting to R15 237 total. How much will he pay back per month if the store offers him the items on a deal of 5.8% interest compounded monthly over a period of two-and-a-half years?

Misconception:

Addressing the misconception:

V. Alex wants to save money to go for a holiday in three years' time. He needs R8 000 for his holiday. He has three options of saving his money:

Option A: At 10% per annum simple interest
Option B: At 3.25% compounded quarterly
Option C: At 7.5% per annum compound interest

Which option will allow Alex to save the least amount of money presently so he can still enjoy his planned holiday in three years' time?

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Misconception: ____

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Addressing the misconception:

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Nominal and Effective Interest Rates

As we have seen from the work covered thus far, very often interest can be compounded more than once a year. Notice that in the examples we looked at where the interest was calculated more than once a year, the interest rate was simply stated as compounded quarterly etcetera, and the words 'per annum' were omitted.

Why do you think this was done?

If the interest is compounded quarterly at 12%, this means that interest was calculated every quarter at 12%. The difference is now if the interest is calculated quarterly at 12% per annum. This implies that for the year the interest is 12%, therefore, it should be compounded at $12 \div 4 = 3\%$ at every quarter.

Let us investigate these concepts which we call nominal and effective interest rates to gain an understanding of them.

Worked Example 3: Nominal and Effective Interest Rates (10 min)

The facilitator will now explain this example to you.

- Remember to make notes as the facilitator is talking and ask as many questions as possible to clarify any misconceptions that may occur.

Example 3:

Suppose that R10 000 is invested at 12% per annum compounded quarterly. The growth of the investment can be tabulated as follows:

Since the interest rate is 12% per annum, the rate at which interest will be calculated per quarter will be 12 ÷ 4 = 3%.

QUARTER PASSED	VALUE OF INVESTMENT
0	10 000
1	$A = P\left(1+\frac{i}{100}\right)^n$ $= 10\,000\left(1+\frac{3}{100}\right)^1$ $= R10\,300$
2	$A = P\left(1+\frac{i}{100}\right)^n$ $= 10\,300\left(1+\frac{3}{100}\right)^1$ $= R10\,609$
3	$A = P\left(1+\frac{i}{100}\right)^n$ $= 10\,609\left(1+\frac{3}{100}\right)^1$ $= R10\,927.27$
4	$A = P\left(1+\frac{i}{100}\right)^n$ $= 10\,927.27\left(1+\frac{3}{100}\right)^1$ $= R11\,255.09$

If we look at the final amount of R11 255.09 we can determine that the investment actually grew by R1 255.09. This equates to a percentage increase of $\frac{1255.09}{10000} \times 100 = 12.5509\%$.

Therefore, it can be seen from this example that the quoted interest rate of 12% is the **nominal interest rate,** and the actual interest rate with which the investment grew in the one-year period was 12.5509%, which is the **effective interest rate.**

Nominal and Effective Interest Rate Calculations

We can calculate the effective interest rate relative to the nominal interest rate by using the following formula:

$$1 + i_{eff} = \left(\frac{1 + i^m}{100 \times m} \right)^m$$

i_{eff} = Effective interest rate

i^m = Nominal interest rate

m = Number of times interest is calculated per annum.

Remember the above formula is just showing a comparison between Nominal and effective interest rates, and the investment period and principle amount is not relevant to this calculation.

Worked Example 4 & 5: Nominal and Effective Interest Rates (10 min)

- With your assistance, the facilitator will now demonstrate some basic calculations.

Example 4:

Determine the effective annual interest rate of 11.8% compounded quarterly.

Solution:

Compounded quarterly means that interest is calculated four times per annum.

$$1 + i_{eff} = \left(1 + \frac{i^m}{100 \times m} \right)^m$$

$$1 + i_{eff} = \left(1 + \frac{11.8}{100 \times 4} \right)^4$$

$$i_{eff} = (1.0295)^4 - 1$$

$$= 0.12332$$

$$= 0.12332 \times 100$$

$$= 12.332\%$$

Example 5:

Determine the nominal interest rate, compounded monthly, which results in an effective interest rate of 12.4%.

Solution:

$$1+i_{eff} = \left(1+\frac{i^m}{100\times m}\right)^m$$

$$1+0.124=\left(1+\frac{i^m}{100\times 12}\right)^{12}$$

$$1.124=\left(1+\frac{i^m}{1200}\right)^{12}$$

$$\sqrt[12]{1.124}-1=\frac{i^m}{1200}$$

$$(1.009788745-1)\times 1200 = i^m$$

$$\therefore i^m = 11.75\%$$

Now that we clearly see the difference between nominal and effective interest rates, it is necessary to modify our previous equations for compound growth. This is done so that learners do not forget that nominal and effective rates must apply in examples wherein the investment period and the principle investment amount are of relevance.

Using the principles we have just investigated, we can modify the compound growth formula to the following:

$$A=P\left(1+\frac{i^m}{100\times m}\right)^{t\times m}$$

Where:

$i^m =$ Nominal interest rate

$m =$ Number of times interest is calculated per annum

$t =$ Time period for the investment

This formula takes into account all of the aspects which the learner is required to know. If the learner uses this formula, he/she will always be prompted by the equation itself and aspects will not be left out, which would cause him/her to lose marks.

Activity 3: Nominal and Effective Interest Rates

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	20 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts.
3. Every group will have an opportunity to provide feedback.

I. Shristi inherited a sum of money which she invested at 11.5% per annum calculated monthly. She kept her money in a fixed investment for a period of 10 years. At the end of the 10th year she had R22 000 in the bank. How much money did Shristi invest initially?

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II. What is the interest on R12 000 in three years at 6% per annum compounded bi-annually?

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Common Errors:

- Many learners find manipulation of the formulae to be a barrier. A simple way is to use variables up until the second-last step and then simply solve for the unknown. Too often learners spend a lot of time trying to manipulate the formulae incorrectly, putting all the correct values into an incorrect formula.
- Learners often forget to take into consideration the nominal and effective interest rates as it sounds too complicated. In fact, these are simple concepts to understand. In addition to this, learners often forget to divide the interest rate by 100. They also confuse

the variable n and use the incorrect values. By using the modified formulae, learners are forced to remember these finer points, which often result in errors.

- Another common mistake is that learners get afraid when the question speaks of more than one transaction. This almost always results in an error, which causes the learners to lose valuable marks. We can easily combat the latter problem by making use of time lines.

Time Lines

Very often people reinvest money that was invested, at different interest rates and for different time periods. Such situations require the compound interest to be calculated more than once. When more than one calculation is required, it is advisable to make use of time lines. Time lines provide a visual of the information represented, in an ordered way, thus ensuring that problems involving multi-step calculations are easier to understand and solve.

Worked Example 6: Time Lines (15 min)

- With your assistance, the facilitator will now demonstrate some basic calculations.

Example 6:

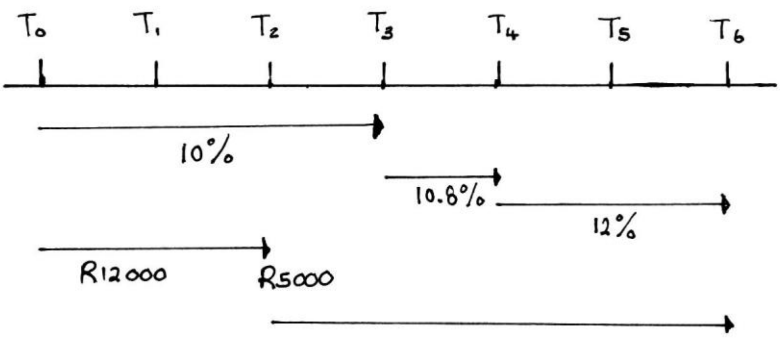
Ramil invested R12 000 into a savings account. He was given an interest rate of 10% per annum compounded quarterly for three years. Two years after the initial investment he deposited a further R5 000 into the savings scheme. The interest rate changed to 10.8% p.a. at the start of the 4th year compounded bi-annually. He kept his money for a further two years at an interest rate of 12% per annum compound interest.

I. Show the time line for the above investment.

II. Determine the value of the investment after 6 years.

Solution:

I.



II. Up to the end of the second year:

$$A = P\left(1+\frac{i^m}{100\times m}\right)^{t\times m}$$
$$= 12\,000\left(1+\frac{10}{100\times 4}\right)^{2\times 4}$$
$$= 12\,000(1.025)^8$$
$$= R14620.83477$$

At the end of the second year Ramil deposited a further R5 000 into the account.

R14 620.83477 + R5 000 = R19 620.83477

T_2 *to* T_3 :

$$A = P\left(1+\frac{i^m}{100\times 4}\right)^{1\times 4}$$
$$= 19620.83477\left(1+\frac{10}{100\times 4}\right)^{1\times 4}$$
$$= 19620.83477(1.025)^4$$
$$= R21657.73034$$

T_3 *to* T_4 :

$$A = 21657.73044\left(1+\frac{10.8}{100\times 2}\right)^{1\times 2}$$
$$= 21657.73044(1.054)^2$$
$$= R24059.91927$$

T_4 *to* T_6 :

$$A = 24059.91927\left(1+\frac{12}{100\times 1}\right)^{2\times 1}$$
$$= 24059.91927(1.12)^2$$
$$= 24059.91927(1.2544)$$
$$= R30180.76$$

At the end of the 6th year Ramil will have R30 180.76 in his account.

NB: The above example has shown us a few very important things that are often found to be a stumbling block for many learners.

- When using time lines, always start with T_0. T_0 is the start of the first year and then T_1 is the end of the first year. T_1 is also the start of the second year, and so forth.

- There was one equation that was used for all calculations (the modified compound growth formula). When there are too many formulas being used, learners tend to become confused and very often use the incorrect formula.

- Intermediate answers are never rounded off as this reduces the accuracy at the end. Only the final answer is rounded off.

- Notice how the accrued amount at the end of a particular time period becomes the principle amount for the next period. This requires understanding of the question and the terminology.

- The easiest way to successfully complete an example of this nature is to break down the scenario into a number of different transactions.

Activity 4: Calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	45 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

Answer the following questions. Show all calculations.

I. How much would an investment of R500 accumulate to in three years if interest were paid at 6% per annum compounded for the first year, and 8% per annum compounded monthly for the next two years?

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II. Zama invests R15 000 for five years. For the first three years it earns interest at 8.5% per annum compounded quarterly, and for the last two years, 7.5% per annum compounded semi-annually.

a) Show the time line for the above investment.

b) What is the value of the investment after four years?

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c) What is the total interest earned over five years?

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III. Ishaan deposited R9 000 into a savings account. He was given an interest rate of 10% compound interest for three years. The interest rate changed to 11.6% per annum compounded quarterly for the next two years. He kept his money for a further two years at 12% per annum compound interest. Two years after the initial investment he deposited a further R4 500 into the savings scheme. Four years after the initial investment he withdrew an amount of R5 250. Show the time line for the following investment and also determine the value of the investment at the start of the 7th year.

IV. Nelly deposited R2 500 into a savings scheme at an interest rate of 8.8% per annum compounded bi-annually for three years. At the end of the 2nd year she deposited R5 000 into the savings scheme. At the end of the 3rd year Nelly withdrew R3 000 and transferred her balance to another scheme that offered her an interest rate of 10.75% per annum compound interest for two years. She moved her investment again and earned a further 7% per annum compounded quarterly for a period of three years.

a) Show the following investment on a timeline.

b) Calculate the value of the investment after six-and-a-half years.

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Simple and Compound Decay

Thus far we have looked at aspects which showed compound or simple growth of an investment. Not all investments will grow in time. Certain investments lose value over time. Examples of these types of investments are the purchase of assets such as motor vehicles and office equipment. This brings us to the concept of depreciation.

Again the terminology that is used in this section is important. It will allow learners to understand what the questions are asking. As an educator, please ensure that your learners always understand the terminology before every section. This allows the subsequent lessons to be more beneficial as they will understand the terminology that is being used.

The two types of decay that are relevant are as follows:

1. Simple decay: also known as **straight line depreciation**
2. Compound decay: also known as **reducing balance depreciation**

It is essential that learners understand the difference between these two types of decay. Let us investigate some simple ways to explain effectively the difference between the two concepts.

Simple Decay: Straight Line Depreciation

Simple decay or straight line depreciation are the terms given to an investment type that loses value over time. The loss of value is calculated as a percentage of the original value. The value per year, for example, will decrease by the same amount. It is theoretically valid that in this type of depreciation there will come a point when the value of a certain item will reach zero. This, in practise, is not very likely, as vehicles or machinery will always have some value.

Let us investigate this type of depreciation through the following example.

Worked Example 7: Simple and Compound Decay (15 min)

Pay careful attention, make notes and ask questions.

Example 7:

Ruvi bought a new car for R300 000. The car depreciates at 12.5% p.a. on a straight line depreciation method. What will the value of Ruvi's car be in five years' time?

Solution:
This can be easily calculated using a simple table:

The car depreciates at 12.5% per annum. Therefore, $\frac{12.5}{100} \times 300\,000 = R37\,500$. This implies that the car will lose R37 500 in value every year.

Age of car in years	Value of car
0	R300 000
1	R300 000 – R37 500 = R262 500
2	R262 500 – R37 500 = R225 000
3	R225 000 – R37 500 = R187 500
4	R187 500 – R37 500 = R150 000
5	R150 000 – R37 500 = R112 500

Notice how the value by which the vehicle depreciates every year remains the same.

It would not be practical to calculate straight line depreciation in this manner. From the example above it can be seen that this is very similar to simple growth, as the amount by which the vehicle depreciates remains the same every year. We then can modify the simple growth formula as follows:

The "+" sign is replaced with a "-"sign, as depreciation means loss of value.

$$A = P(1 - in)$$

In straight line depreciation the variables are as follows:
A = Final depreciated amount
P = Original amount
i = Depreciation percentage
n = Number of calculations of depreciation

The same worked example above:

$$\begin{aligned} A &= P(1 - in) \\ &= 300\,000\left(1 - \left(\frac{12.5}{100} \times 5\right)\right) \\ &= 300\,000(1 - 0.625) \\ &= 300\,000(0.375) \\ &= R112\,500 \end{aligned}$$

Compound Decay: Reducing Balance Depreciation

When dealing with compound decay, the most valuable piece of information the learner can have is that the value of the item each year will depreciate or reduce in value as a percentage of its current value. In this type of depreciation it is impossible for the item to ever reach zero value.

Let us investigate the way reducing balance depreciation or compound decay works.

Activity 5: Compound Decay (10 min)

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	10 min	• Flipchart • Permanent markers	None

In your groups you will:

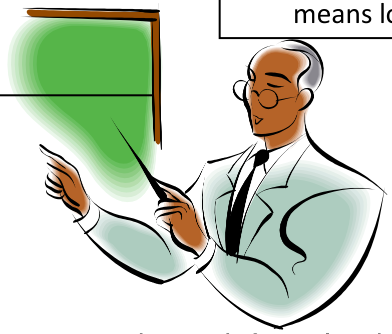
1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are
3. Every group will have an opportunity to provide feedback

Consider the worked example 7 on page 20. Rishi bought the same car as Ruvi at the same time. His car depreciated at 13.8% p.a. on a reducing balance depreciation method. What will the value of Rishi's car be after five years? Show all working.

If need be we can then modify the compound growth formula as follows:

$$A = P\left(1 - \frac{i}{100 \times m}\right)^{t \times m}$$

The "+" sign is replaced with a "-" sign since depreciation means loss of value.



The same example can be calculated using compound growth formula which we will modify:

$$\begin{aligned} A &= P\left(1 - \frac{i}{100 \times m}\right)^{t \times m} \\ &= 300\,000\left(1 - \frac{13.8}{100 \times 1}\right)^{5 \times 1} \\ &= 300\,000(0.862)^5 \\ &= R142\,776.78 \end{aligned}$$

From the above example, the following important points can be deduced:

- The value by which the car decreased was different every year. It was a set percentage of the current value of the car.
- In this form of depreciation, the value of the asset can never reach zero.
- The compound growth formula which we have been using previously has been modified and the plus sign was replaced with a minus sign, as depreciation represents reduction in value.
- Ensure that the learners understand the variables. Most importantly, they must know when the formula will have a minus sign instead of a plus sign. A national diagnostic report of Grade 12 examination results revealed that one of the most common mistakes is the use of the wrong formula. Learners use the compound or simple growth formula instead of changing the sign to minus when dealing with decay situations.

Calculating the Time Period of a Loan or Investment

This section creates many problems for learners. To effectively calculate the time period of a loan or investment, learners need to incorporate the concept of logarithms. There is no need to have a revision of the logarithmic rules, as the section relevant to financial maths is fairly simple and involves few rules from logarithms.

Let us investigate the easiest ways to successfully obtain the period of a loan or investment by means of an example.

Worked Example 8: Simple and Compound Decay (15 min)

Pay careful attention, make notes and ask questions.

Example 8:

Karen deposits R5 000 into a savings account wherein she earns interest at a rate of 9.7% compounded quarterly. How long will it take for Karen to triple her savings?

Solution:

$$A = P\left(1 + \frac{i}{100 \times m}\right)^{t \times m}$$

$$15\ 000 = 5\ 000\left(1 + \frac{9.7}{100 \times 4}\right)^{t \times 4}$$

$$\frac{15\,000}{5\,000} = (1.02425)^{4t}$$

$$3 = (1.02425)^{4t}$$

$$\log 3 = \log(1.02425)^{4t}$$

$$\log 3 = 4t \times \log(1.02425)$$

$$\frac{\log 3}{\log(1.02425)} = 4t$$

$$45.85071208 = 4t$$

$$\therefore t = \frac{45.85071208}{4}$$

$$= 11.46267802$$

$$0.46267802 \times 12 = 5.5$$

$$\therefore \textit{It will take Karen } 11 \textit{ years and } 5\tfrac{1}{2} \textit{ months triple her investment.}$$

Up to this point we have performed the calculation as normal. The problem arises now when we need to solve for t .
Let us now introduce logs on both sides of the equation.

The logarithmic rules allow us to bring the $4t$ down from the exponent position.

The time period is now given as shown alongside. We take the decimal part of the solution and multiply to 12 to get the months.

- As we can see from the above example, the formula that we have used remains the same. We always simply add to or remove from the existing formula. This ensures that learners do not make unnecessary mistakes with formulae.
- We used logarithms in a short and simple way to avoid confusing learners with change of base laws etcetera.
- A recent departmental survey of Grade 12 examination results stated the following: **"*It appears that teaching and learning focuses too much on previous examination papers. This practice compromises conceptual understanding as learners are not exposed to innovative, fresh questions from other sources. They do not learn to read, think, comprehend and then answer the questions. Memoranda with answers***

are readily available from newspapers, the Internet and schools. Learners often do not attempt questions before consulting the memorandum. They simply follow algorithms in the memorandum without much understanding of why and how each step develops."

- We need to ensure that we get our learners to understand the concepts and where or why something is done. It is therefore of utmost importance that we make the understanding and usage of formulae as simple as possible.

Activity 6: Calculating the Time Periods on Loans and Investments

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

Answer the following questions. Show all calculations.

I. A new motor car, which costs R288 000, depreciates on a reducing balance method at a rate of 8.75% per annum. Calculate how long it would take for the car to depreciate to 40% of its current value.

II. Determine how many years it would take for earthmoving equipment to depreciate to a quarter of its original value, considering that earthmoving equipment depreciates on a reducing balance method at a rate of 15.8% per annum.

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III. R8 000 depreciates at 7.4% per annum on a reducing balance method to a value of R2 652.64 over a period of time. What will the value of R8 000 be over the same time period if depreciation was at the same rate on a linear basis?

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IV. Ayesha opens an investment policy for her daughter. She deposits R10 000 and allows the money to grow with interest. After how many years will Ayesha's investment be doubled if the interest rate is at 6.5% per annum compounded semi-annually?

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V. Sahil buys new computer equipment for R160 000. If the equipment depreciates at a rate of 12.8% per annum on a reducing balance basis, after how long will the equipment be worth 30% of the money he invested?

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Annuities

Investopedia defines an annuity as: “A financial product sold by financial institutions which is designed to accept and grow funds from an individual and then, upon annuitization, pays out a stream of payments to the individual at a later point in time. Annuities are primarily used as a means of securing a steady cash flow for an individual during their retirement years.

Annuities can be structured according to a wide array of details and factors, such as the duration of time that payments from the annuity can be guaranteed to continue. Annuities can be created so that, upon annuitization, payments will continue so long as either the annuitant or their spouse is alive. Alternatively, annuities can be structured to pay out funds for a fixed amount of time, such as 20 years, regardless of how long the annuitant lives.

Annuities can be structured to provide fixed periodic payments to the annuitant or variable payments. The intent of variable annuities is to allow the annuitant to receive greater payments if investments of the annuity fund do well and smaller payments if its investments do poorly. This provides for a less stable cash flow than a fixed annuity, but allows the annuitant to reap the benefits of strong returns from their fund's investments.

The different ways in which annuities can be structured provide individuals seeking annuities with the flexibility to construct an annuity contract that will best meet their needs.”

At this level we need to consider two types of annuities:

1. Future Value Annuities
2. Present Value Annuities

Let us investigate the difference between the two types of annuities.

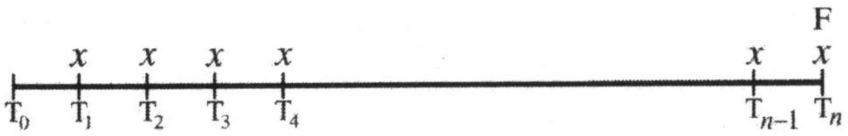
Future Value Annuities

In Future Value Annuities, money or payments are made at regular intervals with an aim to save money for future needs. The interest will allow the investment to grow depending on the rate of interest. Common future value annuities are pension funds, endowment policies, investment policies and sinking funds.

There are several ways in which the calculations can be performed on future value annuities. We can use simple time lines and the formulae to which we are accustomed, or we can use the sum of a geometric series. By exposing the learners to all these methods we are indeed creating a better understanding but also to the weaker learners we are creating a greater misunderstanding. It is important for learners to use the simplest method and that is the method of using the Future Value Annuity Formula, which is derived from the sum of a geometric series.

Let us investigate the derivation of the formula, which is not examinable but shows the learners the validity of the formula.

Derivation of Future Value Annuity Formula:



(Derivation as per Mind action series grade 12)
Using the time line above, we can derive the Future Value Annuity Formula:

$$F = x + x(1+i)^1 + x(1+i)^2 + x(1+i)^3 + ... + x(1+i)^{n-1}$$

$$F = \frac{a\left(r^n - 1\right)}{r-1} \quad ; r \neq 1$$

$$F = \frac{x\left[(1+i)^n - 1\right]}{(1+i)-1}$$

$$\therefore F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

F = Future Value Amount

x = Amount that is paid in at regular intervals

n = Number of payments

i = Interest rate as a decimal

There are several critical points that cause learners to lose marks in this section:

- Confusing the above formula with the Present Value Formula
- Using the incorrect number for n
- Forgetting to divide the interest rate by 100

Let us see how this formula may be applied by means of some examples.

Before we get into the example, it is again essential that we modify the formula so as to ensure minimal error:

$$F = \frac{x\left[\left(1+\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times m}}$$

Worked Example 9: Annuities (15 min)

Pay careful attention, make notes and ask questions.

Worked Example 9:

Jezel wants to invest money in a unit trust. She deposits R750 immediately into the trust and thereafter makes monthly payments at the end of each month into the trust. Interest is set at 7.4% per annum compounded monthly. Calculate the value in the trust at the end of five years.

$$F = \frac{x\left[\left(1-\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times m}}$$

$$\begin{aligned}
F &= \frac{x\left[\left(1-\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times m}} \\
&= \frac{750\left[\left(1+\frac{7.4}{100\times 12}\right)^{61}-1\right]}{\frac{7.4}{100\times 12}} \\
&= \frac{750[0.4550082428]}{\frac{7.4}{100\times 12}} \\
&= R55\ 338.84
\end{aligned}$$

- Remember that the critical part of the calculation is to draw a timeline so as to ascertain the number of payments. The formula will take care of the rest.

Activity 7: Annuities

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	20 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

Mr Mzobe wants to invest money for his daughter's education. When his daughter turns one year old he invests an amount of R1 800 into an investment policy, which earns interest at a rate of 11.2% per annum compounded monthly. He makes payments every month end. After 12 years Mr Mzobe becomes retrenched and stops making monthly payments.

I. How much money is available to his daughter on her 18th birthday?

II. How much more would the investment have been had Mr Mzobe not become retrenched and stopped his monthly payments?

- The most important aspect that learners need to know is the calculation of the number of payments. The modified formula will take care of the rest. Using time lines to evaluate and determine the number of payments is the best option. We should avoid, as teachers, saying to learners for example, "…when you see the terms paid immediately add one to the number of payments…" Although this is true, we should rather ensure our learners understand how it is that the number of payments increases by one rather than just looking for words in the question that tell them to do so.

- The effect of nominal and effective interest rates and the interest rate being divided by 100 are not aspects that learners do not understand, but instead these are always aspects that are forgotten to be included in the formula, hence they lose marks unnecessarily. This is why the modified formula is prompting the learner to remember these vital aspects.

Let us investigate a few more types of examples that are common examination type questions and learners get confused whilst answering them.

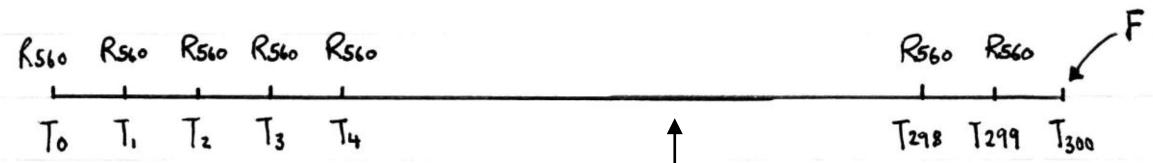
Worked Example 10: Annuities (20 min)

Pay careful attention, make notes and ask questions.

Worked Example 10:

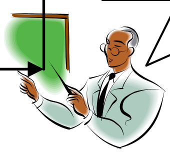
Rowena decides to take out a retirement annuity. She makes payments of R560 into the fund and the payments start immediately. The payments are made in advance, which means that the last payment of R560 is made one month before the annuity pays out. The interest rate for the annuity is 11.5% per annum compounded monthly. Calculate Rowena's payout in 25 years.

Solution:



A total of 300 payments will be made.
$T_0 \rightarrow T_{299} = 300 \; payments$

The first port of call should be to ascertain the number of payments that are made.
Let us use a time line to do this



$$F = \frac{x\left[\left(1+\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times m}}$$

$$= \frac{560\left[\left(1+\frac{11.5}{100\times 12}\right)^{300}-1\right]}{\frac{11.5}{100\times 12}}$$

$$= \frac{560(16.48444003)}{\frac{0.115}{12}}$$

$$= R963\,264.67$$

The value of the annuity will first be calculated at T_{299}. The accumulated amount will gather interest for one month.

$$A = P\left(1+\frac{i}{100\times m}\right)^{t\times m}$$

$$= 963\,264.67\left(1+\frac{11.5}{100\times 12}\right)^{\frac{1}{12}\times 12}$$

$$= 963\,264.67(1.009583333)^{1}$$

$$= R972\,495.96$$

- Again it is shown that the time lines are of utmost importance. Learners should be aware that we cannot use $n = 301$ since there was no 301st payment, but the accumulated amount did gather interest for a further 1 month while no payment was made in the last month.

Activity 8: Effective Teaching Methodology

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to write down other ways which can be used effectively to determine the number of payments, as this showed in the National Diagnostic report to be a major stumbling block for the learners. Share best practice.

Sinking Funds

A sinking fund is a strategic financial plan employed by many companies to ensure they have the funding to replace or upgrade machinery or equipment. A sinking fund is an example of a future value annuity which allows companies to make monthly contributions towards a fund which will accumulate with interest.

Examine the following example and take careful note of all the points that could mislead the learners.

Worked Example 11: Sinking Funds (15 min)

Pay careful attention, make notes and ask questions.

Worked Example 11:

VSP Engineering buys earthmoving equipment to the value of R288 000. The machinery depreciates at a rate of 16.8% per annum on a reducing balance basis. The company will be required in 6 years to upgrade the equipment. The machinery prices inflate by 12.8% per annum effectively. The company starts a sinking fund so as to ensure that the machinery can be upgraded successfully in 6 years. The old machinery is sold and the funds thereof are also used to fund the upgrade. The fund offers an interest rate of 7% per annum compounded monthly.

I. Calculate the value of the existing machinery at the end of 6 years.

II. What will the price on the new upgrades be?

III. How much should the company invest into the sinking fund so as to ensure they will have sufficient funding to replace and upgrade the existing machinery?

Solution:

I. $$A = P\left(1-\frac{i}{100\times m}\right)^{t\times m}$$

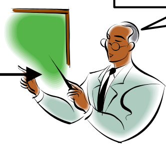
$$= 288\,000\left(1-\frac{16.8}{100\times 1}\right)^{6\times 1}$$

$$= 288\,000(0.832)^6$$

$$= 288\ 000(0.3316957888)$$

$$= R95\,528.39$$

Remember the reducing balance depreciation formula.



II. $$A = P\left(1+\frac{12.8}{100\times 1}\right)^{6\times 1}$$

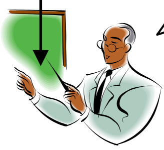
$$= 288\,000(1.128)^6$$

$$= R593\,262.76$$

Straight forward compound growth.

III. Amount required in sinking fund:
= R593 262.76 – R95 528.39
=R497 734.37

The amount of capital required by the company will be the new price of the machinery subtract the depreciated value of the old machinery since it will be sold.



$$F = \frac{x\left[\left(1-\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times m}}$$

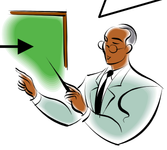
$$497\,734.37 = \frac{x\left[\left(1-\frac{7}{100\times 12}\right)^{72}-1\right]}{\frac{0.07}{12}}$$

$$\left(\frac{0.07}{12}\right)497\,734.37 = x(0.520105504)$$

$$2903.450492 = 0.520105504x$$

$$\therefore x = R5582.43$$

We now use the Future Value Formula and solve for X. Notice how much easier it is to solve for the unknown rather than trying to manipulate the formula.



Quite simple. Now let's take it up a notch. This is one part that always confuses the learner. Take careful note of these aspects.

Activity 7: Sinking Funds

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	20 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

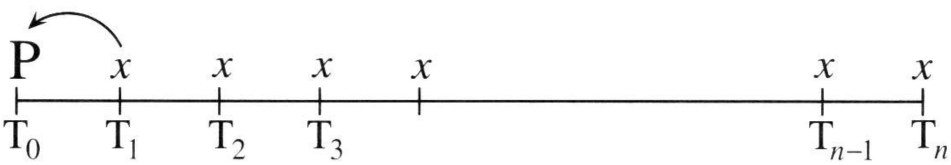
Assuming that VSP engineering decide that from the sinking fund they will draw R4 000 a year to perform basic maintenance on their machinery. How much should they now set aside monthly so as to cover the annual maintenance as well as have enough in the fund to secure new machinery at the end of the 6 year period?

<u>Present Value Annuities</u>

Present value annuities are best described as sums of money that are borrowed from financial institutions and an amount of interest is charged. The loan is completed once the initial capital amount and the interest are paid off completely by means of equal monthly instalments. The interest is calculated on a reducing balance basis. Common examples of present value annuities are loans and mortgages (bonds).

Let us investigate the derivation of the present value annuity formula. Remember this is not directly examinable but learners are required to understand where the formula comes from.

<u>DERIVATION OF PRESENT VALUE ANNUITY FORMULA</u>



$$P = x(1+i)^{-1} + x(1+i)^{-2} + x\left(1+i\right)^{-3} + \ldots + x(1+i)^{n}$$

$$P = \frac{a\left[r^{n} - 1\right]}{r - 1}$$

$$\therefore P = \frac{x(1+i)^{-1}\left[\left\{(1+i)^{-1}\right\}^{n} - 1\right]}{(1+i)^{-1} - 1} \qquad \left[a = x(1+i)^{-1}\ ;\ r = (1+i)^{-1}\right]$$

$$P = \frac{x\left[\left(1+i\right)^{-n} - 1\right]}{(1+i)^{0} - (1+i)^{1}}$$

$$= \frac{x\left[\left(1+i\right)^{-n} - 1\right]}{1 - (1+i)}$$

$$\therefore P = \frac{x\left[\left(1+i\right)^{-n} - 1\right]}{-i}$$

$$\therefore P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}$$

- Remember that the formula for P can only be used if there is a gap between the initial loan and the commencement of the repayment.
- If there is a loan and there is immediate payment then that payment can be deemed to be seen as a deposit and must be subtracted from the initial amount before the formula for P can be applied.
- Learners commonly confuse the formula between Present and Future value Annuities.

- Let us look at a few examples involving the most common examination type questions based on present value annuities. Before we do the examples let us again modify the Present Value Annuity formula so as to minimise errors.

$$P = \frac{x\left[1 - \left(\frac{i}{100 \times m}\right)^{-n}\right]}{\frac{i}{100 \times m}}$$

- P = Present value of the loan
- $x =$ Equal monthly instalments
- i = Interest rate

- n = Number of payments

Worked Example 12: Present Value Annuities (15 min)

Pay careful attention, make notes and ask questions.

Worked Example 12:
Sanam takes out a loan of R250 000 to buy a new car. The loan is taken over 6 years at an interest rate of 10.5% per annum compounded monthly. Calculate:

I. Her monthly instalments if her first payment is one month after she had taken out the loan?
II. Her monthly instalments if she paid an amount of R32 000 immediately and thereafter paid equal monthly instalments?

Solution:

I.

$$P = \frac{x\left[1-\left(1+\frac{i}{100\times m}\right)^{-n}\right]}{\frac{i}{100\times m}}$$

There was a one month gap between the first payment and the loan. Use the formula for *P*.

$$250\,000 = \frac{x\left[1-\left(1+\frac{10.5}{100\times 12}\right)^{-72}\right]}{\frac{10.5}{100\times 12}}$$

$$\left(\frac{0.105}{12}\right)250\,000 = x(0.4659467486)$$

$$\therefore x = R4694.74$$

II. $$P=\frac{x\left[1-\left(1+\frac{i}{100\times m}\right)^{-n}\right]}{\frac{i}{100\times m}}$$

In this example there was no gap between the first payment and the loan. We treat the immediate payment as a deposit and it must be subtracted from the total loan amount before we apply the formula.

$$250\,000-32\,000=\frac{x\left[1-\left(1+\frac{10.5}{100\times 12}\right)^{-72}\right]}{\frac{10.5}{100\times 12}}$$

$$\left(\frac{0.105}{12}\right)218\,000=x(0.4659467486)$$

$$\therefore x=R4093.82$$

Activity 8: Present Value Annuities

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	20 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

Shanal borrows R600 000 from the bank and pays the loan back by means of a monthly instalment of R7 500 commencing one month after the loan was taken. The interest charged on the loan is 14.4% per annum compounded monthly. Determine:

I. How many payments of R7 500 are made?

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II. What is the final amount that Shanal will pay?

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We can also use the present value annuity formula to help calculate the value of a loan at any given time.

This is a common examination type question and learners always get confused when answering these types of questions. Let us investigate by means of an example how these questions should be answered.

Worked Example 13: Balance outstanding on a loan (20 min)

Pay careful attention, make notes and ask questions. This was a previous year question which was answered very badly.

Worked example 13:

A father decided to buy a house for his family for R800 000. He agreed to pay monthly instalments of R10 000 on a loan which incurred interest at a rate of 14% per annum compounded monthly. The first payment was made at the end of the first month.

I. Show that the loan will be paid off in 234 months.

II. Suppose the father encountered unexpected expenses and was unable to pay any instalments at the end of the 120^{th}, 121^{st}, 122^{nd} and 123^{rd} months. At the end of the 124^{th} month he increased his payment so as to still pay off his loan in 234 months by 111 equal monthly payments. Calculate the value of this new instalment.

Solution: (P.T.O.)

I.

$$P_v = \frac{x\left[1-\left(1+\frac{i}{100\times m}\right)^{-n}\right]}{\frac{i}{100\times m}}$$

$$800\ 000 = \frac{10\,000\left[1-\left(1+\frac{14}{100\times 12}\right)^{-n}\right]}{\frac{0.14}{12}}$$

$$\left(\frac{0.14}{12}\right)800\,000 = 10\,000\left[1-(1.011666667)^{-n}\right]$$

$$\frac{9333.333}{10\,000} = 1-(1.011666667)^{-n}$$

$$0.933333333-1 = -(1.011666667)^{-n}$$

$$0.0666666667 = (1.011666667)^{-n}$$

$$\frac{\log(0.0666666666667)}{\log(1.011666667)} = -n$$

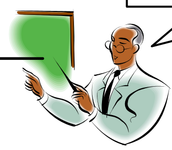
$$-233.47 = -n$$

$$\therefore n = 234$$

We use the present value formula. Always be careful when working with logs. Remember the log of a negative number is undefined.

II. Balance after 119th month:

The balance at the end of the 119th month is the interest + the capital for 119 months minus all the payments made thus far.



$$P = \left(1+\frac{i}{100\times m}\right)^{n} - \frac{x\left[\left(1+\frac{i}{100\times m}\right)^{n}-1\right]}{\frac{i}{100\times 12}}$$

$$= 800\,000\left(1+\frac{14}{100\times 12}\right)^{119} - \frac{10\,000\left[\left(1+\frac{14}{100\times 12}\right)^{119}-1\right]}{\frac{0.14}{12}}$$

$$= R629\ 938.11$$

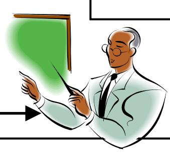
Total payable at the end of the 123rd month:

$$A = P\left(1+\frac{i}{100\times m}\right)^{n}$$

$$= 629\ 938.11\left(1+\frac{14}{100\times 12}\right)^{4}$$

$$= R659\ 853.68$$

Due to the missed payments the balance at the end of 119th month will accumulate interest for the four months when no payment was received. The new instalment is then calculated using this new balance at the end of the 123rd month.



New instalment:

$$A = \frac{x\left[1-\left(1+\frac{i}{100\times m}\right)^{-n}\right]}{\frac{i}{100\times m}}$$

$$659853.68 = \frac{x\left[1-\left(1+\frac{14}{100\times 12}\right)^{-111}\right]}{\frac{0.14}{12}}$$

$$\left(\frac{0.14}{12}\right)659\,853.68 = x(1-(1.011666667)^{-111})$$

$$7698.2929 = 0.7240413146x$$

$$\therefore x = \frac{7698.2929}{0.72404131}$$

$$= R10\,632.39$$

Activity 9: Present Value Annuities			
Group organisation:	**Time:**	**Resources:**	**Appendix:**
Groups of 6	20 min	• Flipchart • Permanent markers	None

In your groups you will:

1. Select a scribe and a spokesperson for this activity only – should rotate from activity to activity.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions are.
3. Every group will have an opportunity to provide feedback.

Senzo takes out a twenty year loan of R250 000. He repays the loan by equal monthly instalments commencing FOUR months after he had taken out the loan. The interest rate is 21% per annum compounded monthly.

I. Calculate the amount owing three months after the loan was granted

II. Senzo was unable to pay the loan from the 31^{st} to the 39^{th} instalment. He made up by arranging a new instalment with the institution. Calculate the new instalment.

III. How much was still owed on the loan after the 163^{rd} payment?

Conclusion

Financial maths requires the learners to have an understanding rather than looking for key phrases or words. It is a relatively easy section to score marks in, and we should strive to ensure our learners achieve maximum results in this section.

According to the diagnostic report by the department of education, these were the common faults that were picked up in the examination:

1. "... many candidates confused the formulae they had to use. Language could also have been a contributing factor to the weaker-than-expected performance in this question."

2. "Many candidates clearly could not differentiate between future value and present value in setting up correct equations. Manipulation errors occurred, which resulted in the application of logarithms to negative numbers, which is clearly not valid. These

candidates, however, ignored the negative sign and calculated the answers."

3. "It appears as if many teachers did not show the learners how the F_v and P_v formulae are derived and how to apply them in real-life situations. This could be the reason for the candidates' lack of clarity in applying the formulae correctly."

4. "Candidates did not understand how each formula works; they tend to associate a formula with a certain form of phrasing in the question. It appears that learners might have been misled in the teaching process as it appeared that once they saw the word 'invest' in the question, they thought it meant the use of the future value formula. This must be corrected and must be taught with understanding."

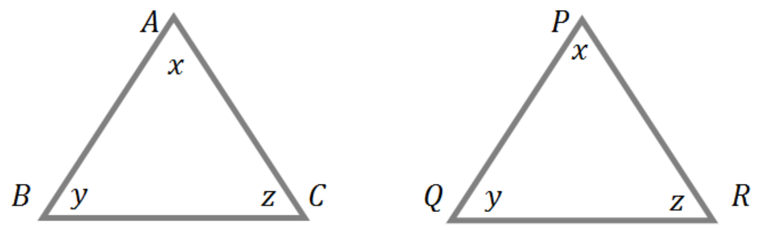
5. "Making a variable the subject of the formula and reading in context must be emphasised."

Module 5:
Sub-topic 1: Similarity-triangles and similarity

Grade 10	Grade 11	Grade 12
1. Revise basic results established in earlier grades regarding lines, angles and triangles, especially the similarity and congruence of triangles. 2. Investigate line segments joining the midpoints of two sides of a triangle.	No similarity.	1. Revise earlier work on the necessary and sufficient conditions for polygons to be similar. 2. Prove (accepting results established in earlier grades): • That a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Mid-point Theorem as a special case of this theorem) • That equiangular triangles are similar • That triangles with sides in proportion are similar, and • The Pythagorean Theorem by similar triangles.

Introduction:

For triangles to be similar means they have the same shape but not the same size.
The conditions for triangles to be similar:
I. Corresponding angles must be equal.

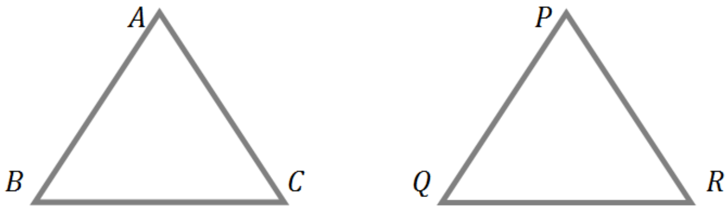


For Δ^s ABC and PQR above:

$\hat{A}=\hat{P}$, $\hat{B}=\hat{Q}$ and $\hat{C}=\hat{R}$

$\therefore \Delta$ ABC III Δ PQR (*LLL*)

II. Corresponding sides must be in the same proportion.



For Δ^s ABC and PQR above:

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} \quad then$$

∴Δ ABC III Δ PQR (*SSS*)

Remember the notation: the side named by the first two letters of the first triangle, AB, corresponds with the side indicated by the first letters, PQ, of the second triangle, etc.

You must be able to prove the following two theorems:

- If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar [IIIΔ^s]
- If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar) [sides of Δ in proportion]

Ratios and Proportion:

- If two ratios are equal then we say they are in the same proportion
  e.g. $\frac{3}{4} = \frac{9}{12}$ we say that 3 and 4 are in the same proportion as 9 and 12

The Midpoint Theorem:

- The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side [Midpoint Theorem]

Misconceptions:

- Do not confuse similarity and congruency
- Congruent triangles have the same shape and size
- Similar triangles have the same shape but not the same size

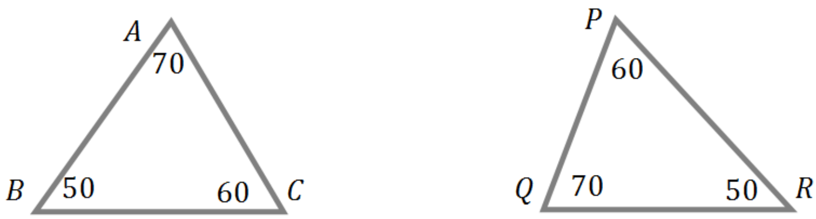
Worked Examples 1, 2 and 3

The facilitator will now explain these examples to you. Remember to make notes as the facilitator is talking and ask as many questions as possible so as to clarify any misconceptions that may occur.

Example 1

Are the flowing pairs of triangles similar or not? Give reasons

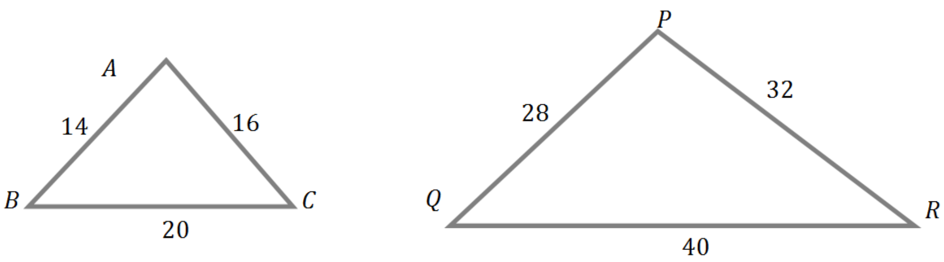
I.



Δ ABC _____ Δ PQR Reason ____________

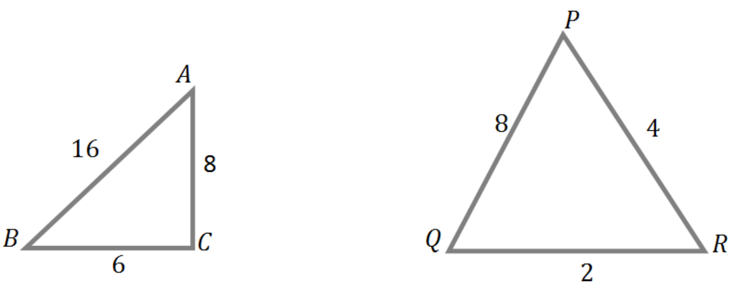
(Be sure to use the correct order of notation)

II.



Δ ABC_____ Δ PQR Reason ___________

III



Δ ABC_ Δ PQR Reason ___________

Solution:

I. ∆ *RST* III ∆ *QRP* reason: *LLL*

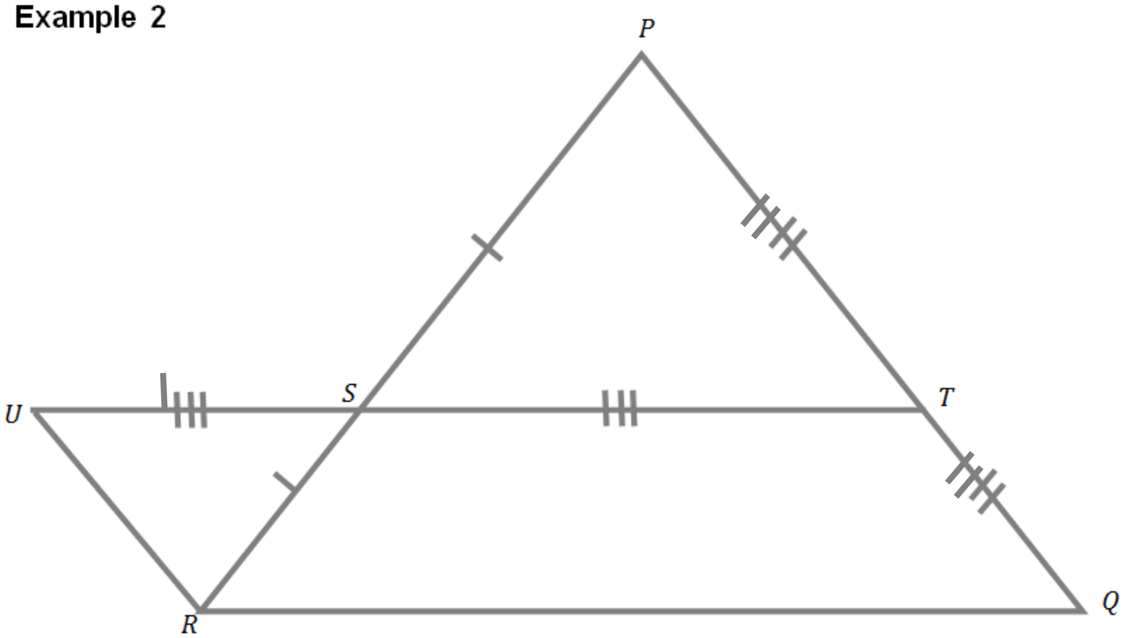
II. ∆ *ABC* III ∆ *QRP* reason: sides of ∆ in proportion

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} \; ; \frac{14}{28} = \frac{20}{40} = \frac{16}{32}$$

III. ∆ *FGH* ~~III~~ ∆ *KLM* reason: sides of triangle not in proportion

$$\frac{16}{8} \neq \frac{6}{12}$$

Example 2



In the diagram S and T are the midpoints of PR and PQ.

TS is produced to U such that ST = SU.

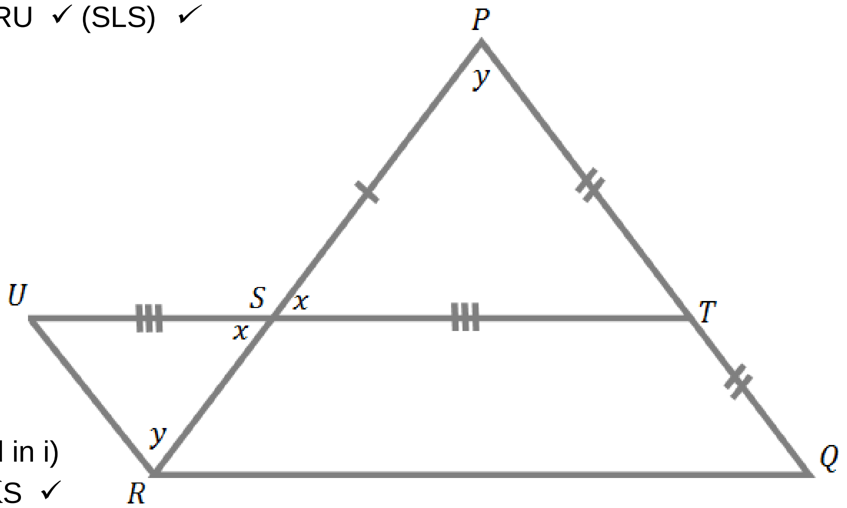
Prove that:

i. ∆ SPT = ∆ SUR
ii. UR II PQ
iii. UTQR is a parallelogram
iv. ST = ½ RQ

Solution:

i. In ∆ SPT and ∆ SRU ✓
- PS = SR (given) ✓
- ST = US (given) ✓
- P$\hat{S}$T = U$\hat{S}$R ✓ (vertically opposite angles) [vert opp *Ls =]*

- ∴ Δ SPT ≡ Δ SRU ✓ (SLS) ✓



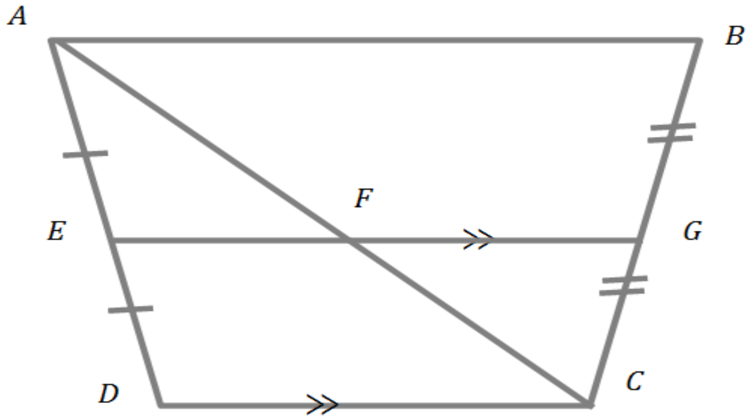
ii. $\hat{P} = U\hat{R}S$ ✓ (Proved in i)
- $\hat{P}$ = alternate $U\hat{R}S$ ✓
- ∴ UR II PQ ✓ (alternate angles are equal) [alt *Ls* =]

iii. UR = PT ✓ (Proved in i)
- = TQ ✓ (T is midpt PQ)
- but UR II TQ ✓ (Proved in ii)
- ∴ UTQR is a parallelogram [pair of opp sides = and II) ✓

Tip: Fill everything in on the figure as you go

iv. UT = RQ ✓ (opp sides of II^{m})
- ST = ½ UT ✓ (given)
- ∴ ST = ½ RQ ✓

Example 3

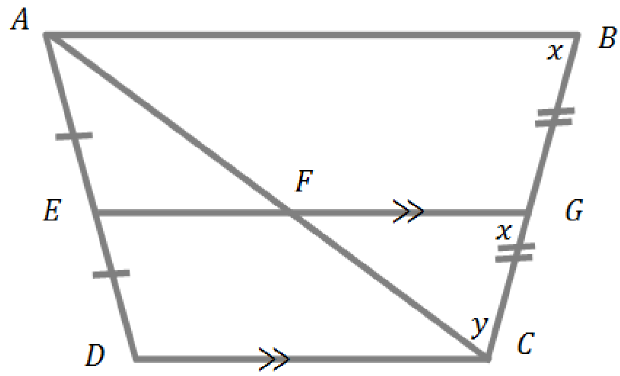


In the figure *EGDC* is a trapezium. *EG* II *DC* ; *DE* = *EA* ; *BG* = *GB*

AC cuts *EG* in *F*

Prove that:

i. *AF = FC*
ii. *FG* II *AB*
iii. Δ *FCG* III Δ *ACB*



Solution:

i. In Δ ADC (always name the Δ) ✓
   - AE = ED (given, E midpt AD) ✓
   - EF II DC (given) ✓
   - ∴ AF = FC (Midpt Theorem) ✓

ii. In Δ ACB
   - AC = FC (proved in i)
   - BG = GC (given, G midpt BC)
   - ∴ FG II AB (Midpt Theorem)

In Δ FCG and Δ ACB
   - $\hat{C} = \hat{C}$ (common, in both Δ^s) ✓
   - $C\hat{G}F = \hat{B}$ (correp Ls ; FG II AB) ✓
   - ∴ Δ FCG III Δ ACB ✓ (LLL) ✓

Activity 1-3: Basic proofs and calculations			
Group organisation:	**Time:**	**Resources:**	**Appendix:**
Groups of 6	25 min	• Flip chart paper • Whiteboard markers	None

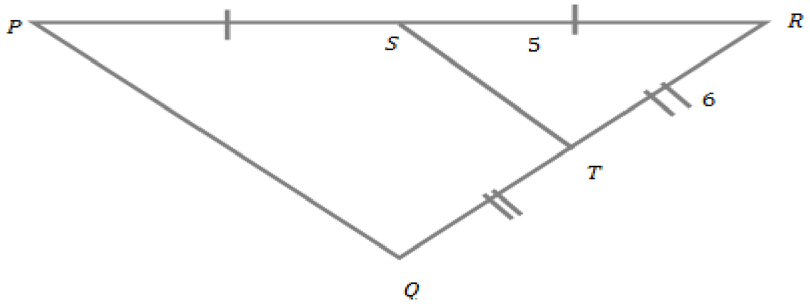
In your groups you will:
1. Select a scribe and a spokesperson for this activity only.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions could be.
3. Every group will have an opportunity to provide feedback.

Activity 1:

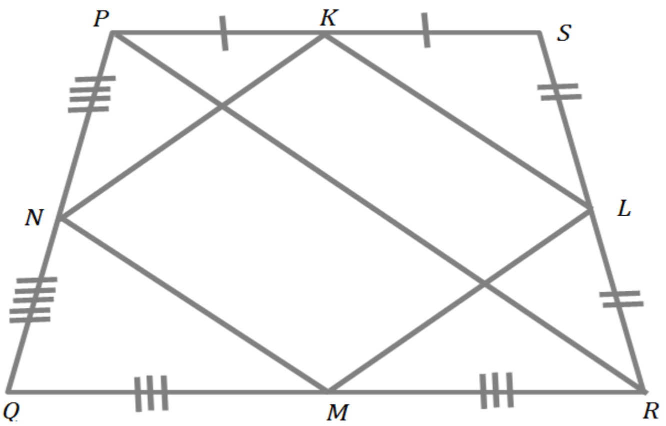
In the diagram SR = 5cm ST = 7cm and TR = 6cm.

S and T are the midpoints of PR and QR respectively. QT = TR

Calculate the perimeter of PQTS. (Give reasons)



Activity 2



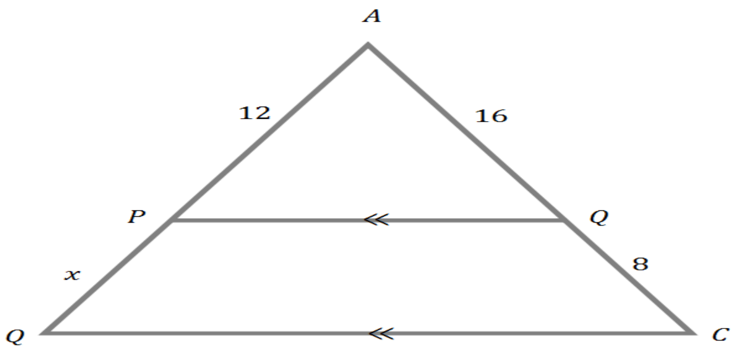
In quadrilateral PQRS, K, L, M and N are midpoints of PS, SR, QR and PQ respectively.

Prove

i. KL II NM
ii. KL = NM
iii. ∆ QNM III ∆ QPR

Activity 3

i.

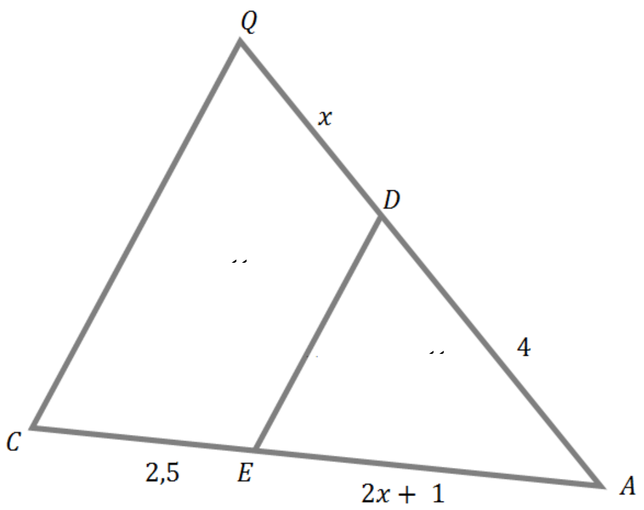


In ∆ ABC, PQ III BC and AP = 12cm, AQ = 16cm and QC = 8cm.

PB = x

Find the value of x. Give reasons.

ii.



In Δ ABC BC II DE BD = x DA= 4CM CE = 2,5CM EA = 2x + 1

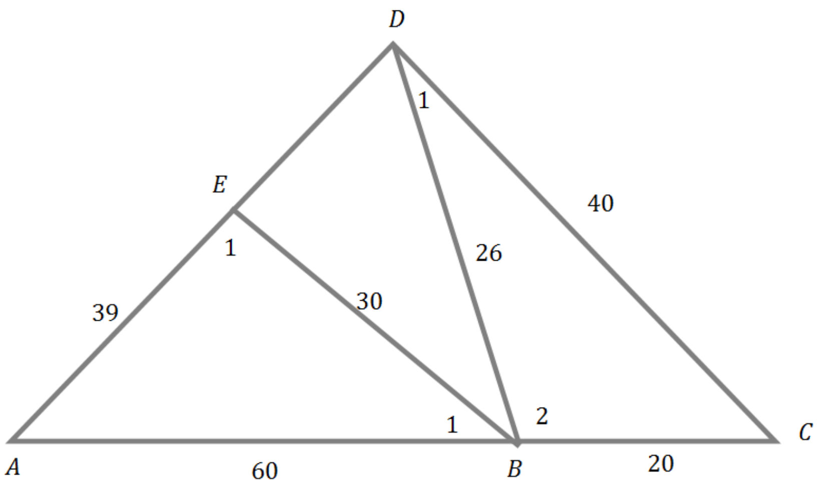
Find the value of x. Give reasons

b) Calculate the length of EA

Worked Examples 4-6

The facilitator will now provide you with more suitable examples to help you understand this concept. Please make notes as the facilitator is talking and don't forget to ask as many questions as possible so as to clarify any misconceptions that may occur.

Example 1



Given Δ DBC with DC = 40 mm. E is a point on DA so that AE = 39mm.
B is a point on AC so that AB = 60mm and BC = 20mm.
EB = 30mm and DB = 26mm.

Prove that

i. Δ AEB III Δ BDC
ii. EB II DC

Solution:

i. In Δ AEB and Δ BDC

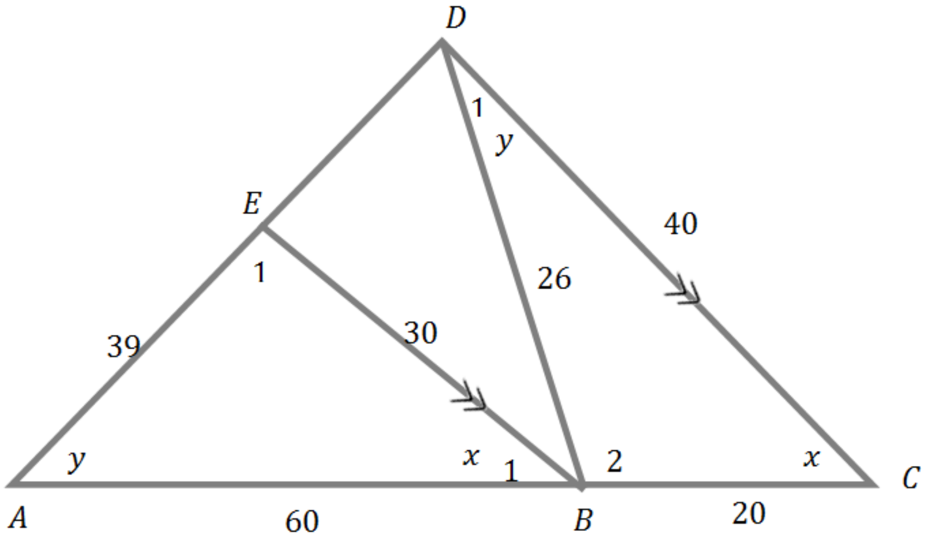
- $\frac{AB}{DC} = \frac{60}{40} = \frac{3}{2}$ ✓ (longest sides)
- $\frac{EB}{BC} = \frac{30}{20} = \frac{3}{2}$ ✓ (shortest sides)
- $\frac{EA}{BD} = \frac{39}{26} = \frac{3}{2}$ ✓ (third pair of sides)

(Because the ratios are equal we can conclude that the Δ^3 are similar)

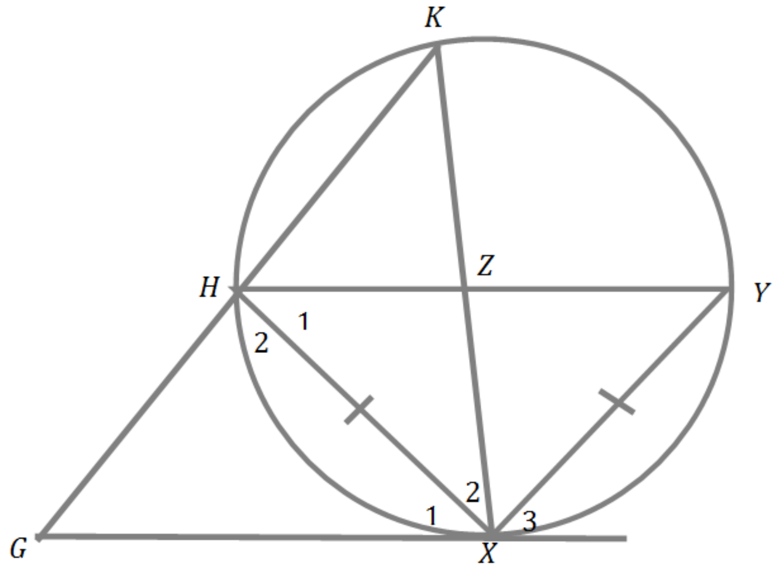
- $\therefore \frac{AB}{DC} = \frac{EB}{BC} = \frac{EA}{BD}$ ✓
- $\therefore$ Δ EAB III Δ BDC ✓ (sides of Δ in prop)

ii. In Δ AEB and Δ BDC

- $A\hat{E}B = \hat{C}$ ✓ [III Δ^s proved in 1]
- but $E\hat{B}A$ is corresponding to $\hat{C}$ ✓
- $\therefore$ EB III DC ✓ (corresp *Ls* =)



Example 2



In the figure GX is a tangent. HX = XY

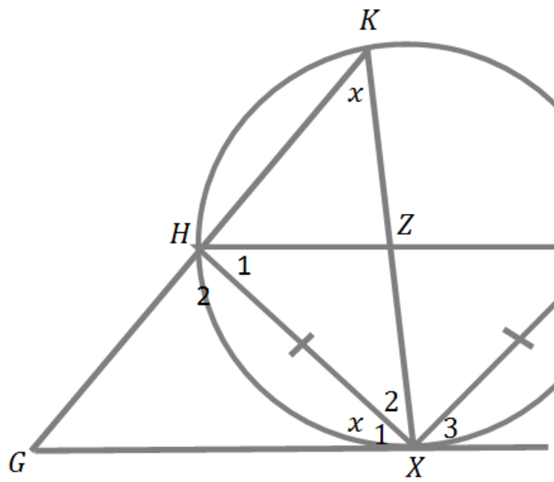
Prove that

i. Δ HGX III Δ XGK
ii. GX^2 = GH.GK

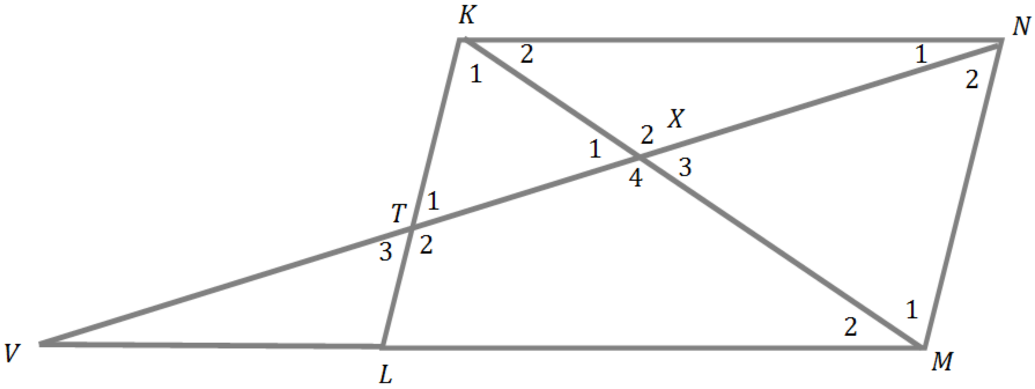
Solution:

i. In Δ HGK and Δ XGK
   - $\hat{X}_1 = \hat{K}$ (tan chord theorem) ✓
   - $\hat{G} = \hat{G}$ (common in both Δ^s) ✓∴ Δ HGX III Δ XGK ✓ (LLL) ✓

ii. Δ HGK III Δ XGK (proved in i)
   - ∴ $\frac{HG}{GX} = \frac{GX}{GK}$ ✓
   - ∴ *$GX^2 = \Delta HG.GK$* ✓ *(cross multiplication)*



Example 3



KLMN is a parallelogram.
T is on KL.
NT produced meets ML produced at V and NT intersects KM at X.

Prove that

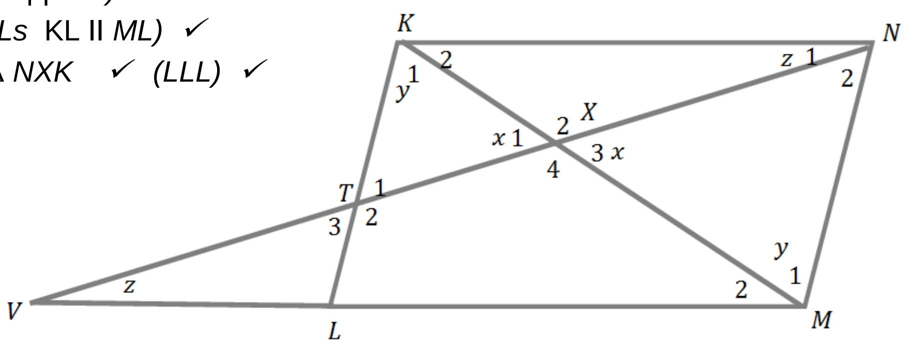
i. $\frac{NX}{XT} = \frac{MX}{XK}$
ii. Δ VXM III Δ NXK
iii. $NX^2 = XT.XV$

Solution:

To prove $\frac{NX}{XT} = \frac{MX}{XK}$ identify 2 Δ^s. One with sides NX and MX, the other with sides XT and XK. Highlight the sides and shade the 2 Δ^s.

i. In Δ NXM and Δ TXK
- $\hat{X}_3 = X_1$ *(vert. opp Ls)* ✓
- $\hat{M}_1 = \hat{K}$ (alt *Ls* KL II MN) ✓
- Δ NXM III Δ *TXK* ✓ *(LLL)* ✓
- $\therefore \frac{NX}{XT} = \frac{MX}{XK}$ ✓

ii. In Δ VXM and Δ NXK
- $\hat{X}_4 = X_2$ (vert. opp *Ls)* ✓
- $\hat{V} = \hat{N}$ (Alt. *Ls* KL II *ML)* ✓
- Δ *VXM* III Δ *NXK* ✓ *(LLL)* ✓



iii. $\frac{NX}{XT} = \frac{MX}{XK}$ ✓ (proved in i)

- ∴ NX $= \frac{MX.XT}{XK}$ ✓
- ∆ VXM III ∆ NXK ✓ (proved in ii)
- ∴ $\frac{NX}{VX} = \frac{XK}{XM}$ ✓
- ∴ NX = $\frac{XK.VX}{XM}$ ✓
- ∴ NX .NX = $\frac{MX.XT}{XK}$ x $\frac{XK.VX}{XM}$ ✓
- NX2 = XT. VX ✓

Activity 4: Proofs and calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	25 min	• Flip chart paper • Whiteboard markers	None

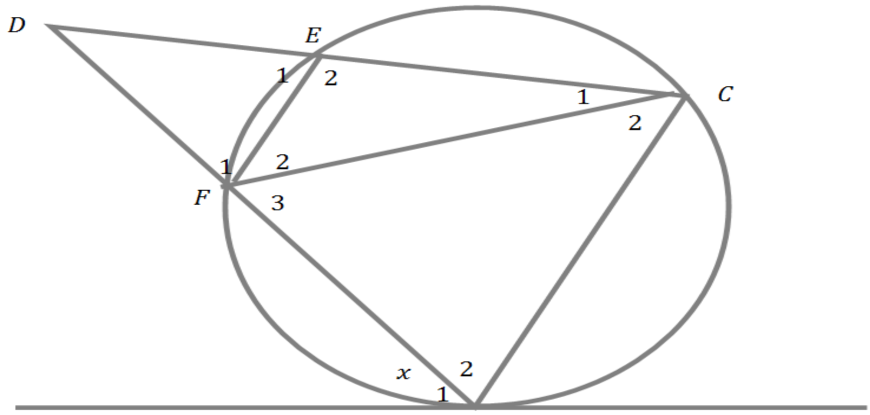
In your groups you will:

1. Select a scribe and a spokesperson for this activity only.(Should rotate from activity to activity)
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions could be.
3. Every group will have an opportunity to provide feedback.

Activity 4:

In the diagram AB is a tangent at B. DC II AB. D$\hat{B}$A = x

I. State, with reasons, two other angles each equal to x.
II. Prove ∆ DEF III CBF
III. Prove DE.DC = DF.DB (Hint. First prove two triangles similar)



Conclusion

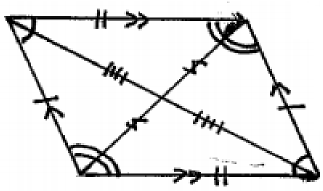
Hint: Use a nice big diagram on which you can mark off, using coloured pencils or highlighters, all the given information, e.g. equal sides, parallel lines, right angles, equal angles, radii, etc.

Sub-topic 2: Quadrilaterals and Cyclic Quadrilaterals

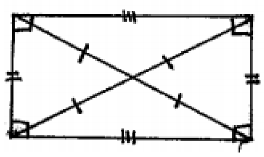
Grade 10	Grade 11	Grade 12
• Define the following special quadrilaterals ○ Kite ○ Parallelogram ○ Rectangle ○ Rhombus ○ Square ○ Trapezium • Investigate and make conjectures about the properties of the sides, angles, diagonals and areas of these quadrilaterals. • Prove these conjectures	• Accept results established in earlier Grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. • Then investigate and prove the theorems of the geometry of circles: ○ The opposite angles of a cyclic quadrilateral are supplementary.	• Revise earlier work

Introduction

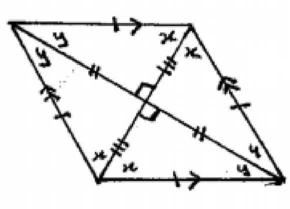
In Grade 10 all the properties of the following quadrilaterals have been investigated and proved.



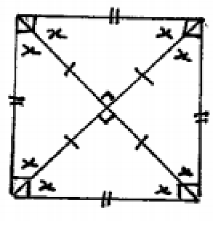
parallelogram



rectangle

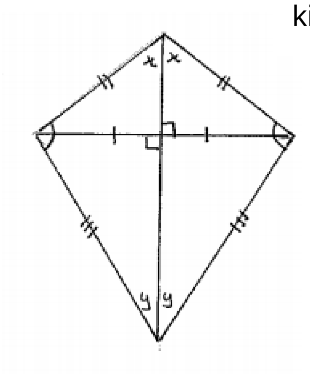


rhombus

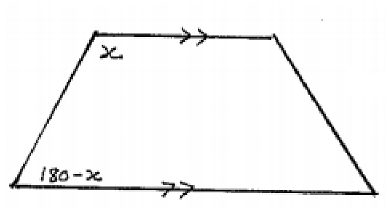


square

kite

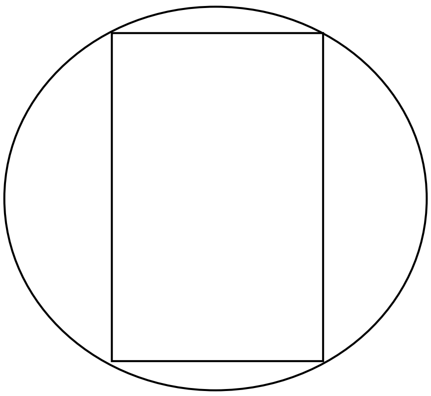


trapezium



In Grade 11 the cyclic quadrilateral is studied

A quadrilateral is cyclic if all four of its vertices lie on the circumference of a circle.



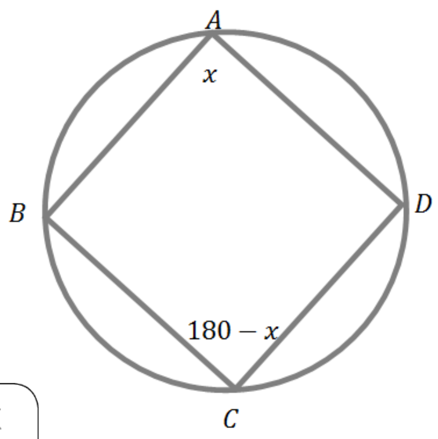
You must be able to prove the following theorem:

- The opposite angles of a cyclic quadrilateral are supplementary.

[opp L^s of a cyclic quad]

$\hat{A} + \hat{C} = 180°$ and

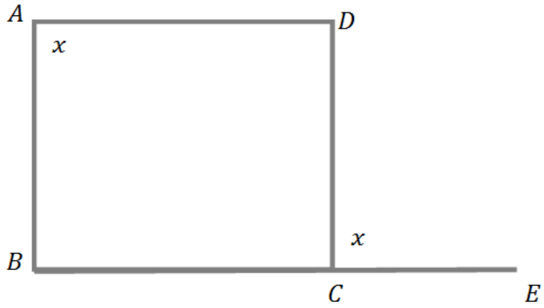
$\hat{B} + \hat{D} = 180°$77



Tip: Pages 9, 10 and 11 of the Examination Guidelines Grade 12, 2014, give a list of all the acceptable reasons for Euclidean Geometry as well as how to abbreviate the theorem.

Corollaries derived from this theorem are necessary in solving riders, e.g.

- If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic [converse opp $\angle^s$ of cyclic quad] OR [opp $\angle^s$ quad sup]
- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle [ext $\angle$ of cyclic quad]



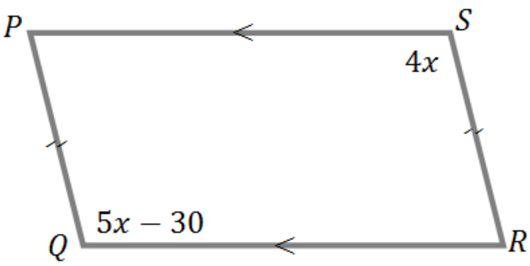
$B\hat{A}D = D\hat{C}E$

- If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic. [ext $\angle$ = int opp] OR [converse ext $\angle$ of a cyclic quad]

Worked Examples 1-4

The facilitator will now provide you with more suitable examples to help you understand this concept. Please make notes as the facilitator is talking and don't forget to ask as many questions as possible so as to clarify any misconceptions that may occur.

Example 1



PQRS is a parallelogram. $\hat{Q} = 5x - 30°$ and . $\hat{S} = 4x$

Calculate the magnitude of

I. $\hat{Q}$ (give reasons)
II. $\hat{P}$ (give reasons)

Solution:

I. $\hat{Q} = \hat{S}$ (opp $\angle^s$ II^m =)
- $\therefore 5x - 30 = 4x$
- $5x - 4x = 30$
- $x = 30°$

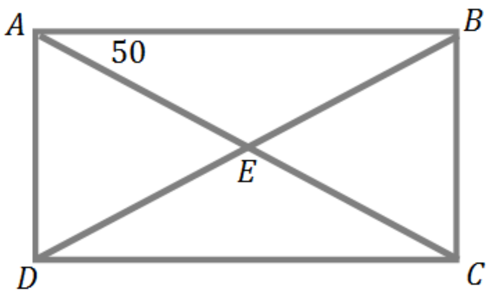
$\therefore$ $\hat{Q}$ = $5x - 30$

- = 5 x 30 – 30
- =150 – 30
- $\hat{Q}$ = 120°

II. $\hat{P} + \hat{Q} = 180°$ [co-int L^s; PS II QR]

- $\hat{P}$ + 120 = 180°
- $\hat{P}$ = 180 -120
- = 60°

Example 2

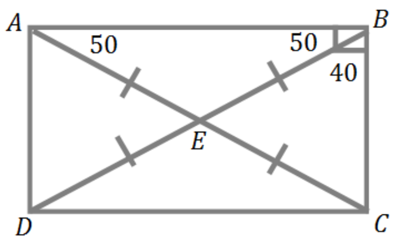


ABCD is a rectangle.
CÂB = 50°
Calculate the size of D$\hat{B}$C. (give reasons)

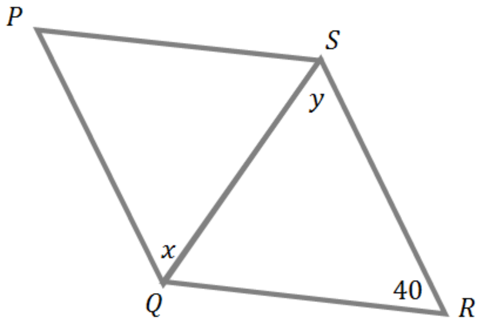
Solution:

In Δ AEB, AE = EB. [diag of rect. bisect at E. diag are equal]

- A$\hat{B}$E = 50° [L^s opp equal sides]
- but $\hat{B}$ = 90° [ABCD rectangle]
- D$\hat{B}$C = 90 – 50
- =40



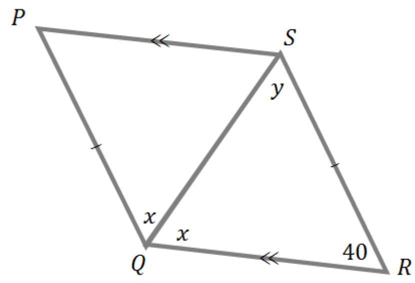
Example 3



PQRS is a rhombus. $\hat{R}$ = 40 P$\hat{Q}$S = x Q$\hat{S}$R = y

Calculate, giving reasons, the values of x and y.

Solution:



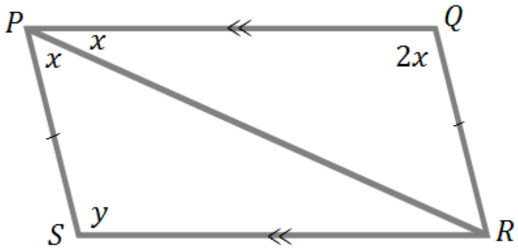
S$\hat{Q}$R = x (diagram of rhombus)

- Q + $\hat{R}$ = 180° [co-interior L^s; PQ II SR]
- 2 x + 40 = 180
- 2 x = 140
- x = 70

Q$\hat{S}$R = P$\hat{Q}$S [alt L^s; PQ II SR]

- y = x = 70
- y = 70°

Example 4



PQRS is a quadrilateral. PS II QR and PQ II SR

- Find the values of x and y. Give reasons
- Name the quadrilateral. Give reasons

Solution:

Q$\hat{P}$S + P$\hat{Q}$R = 180° [co-int L^s; PS II QR]
2 x + 2 x = 180°
4 x = 180°
x = 45°

Q$\hat{P}$S + P$\hat{S}$R = 180° [co-int L^s; PQ II SR]
2 x + y = 180°
2 x 45 + y = 180
y = 180 - 90
= 90°
PQRS is a square. Interior angles = 90° and diagonal PR bisects Q$\hat{P}$S

Activity 1: Proofs and calculations

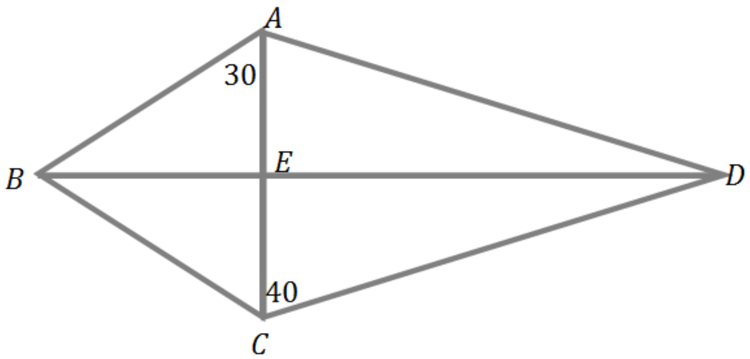
Group organisation:	Time:	Resources:	Appendix:
Individual	5 min	• None	None

In your groups you will:

1. Complete the following calculations and write your answers in the space provided.

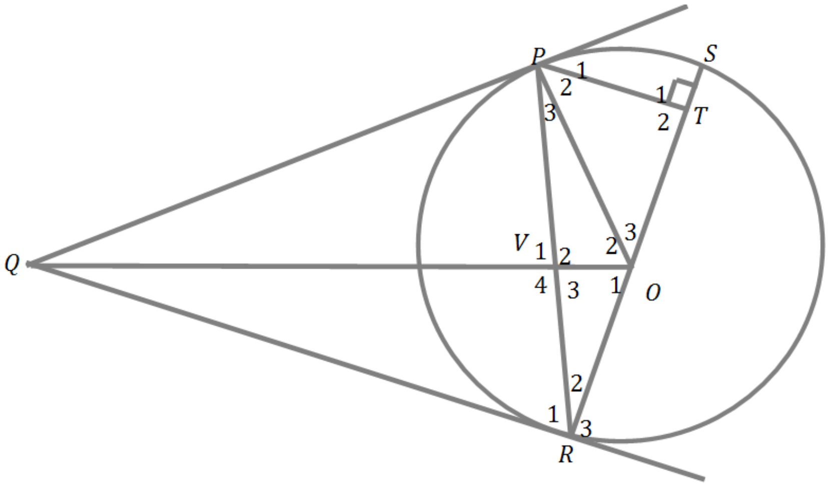
ABCD is a kite with $B\hat{A}C = 30°$ and $A\hat{C}D = 40°$
Calculate the size of:

I. $E\hat{D}C$
II. $C\hat{B}D$



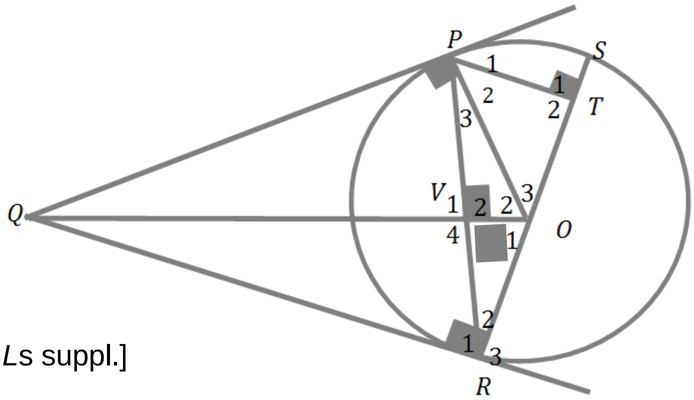
Worked Examples 5-6

The following examples will further help you understand this concept and how to explain this easily to your learners. Please make notes as the facilitator is talking and don't forget to ask as many questions as possible so as to clarify any misconceptions that may occur.



P and QR are tangents to the circle. O is the centre of the circle.
SR is a diameter. PT⊥ SR. QO intersects PR at V.
Prove:

I. PQRO is a cyclic quadrilateral
II. PQRO is also a kit
III. PTOV is a cyclic quadrilateral



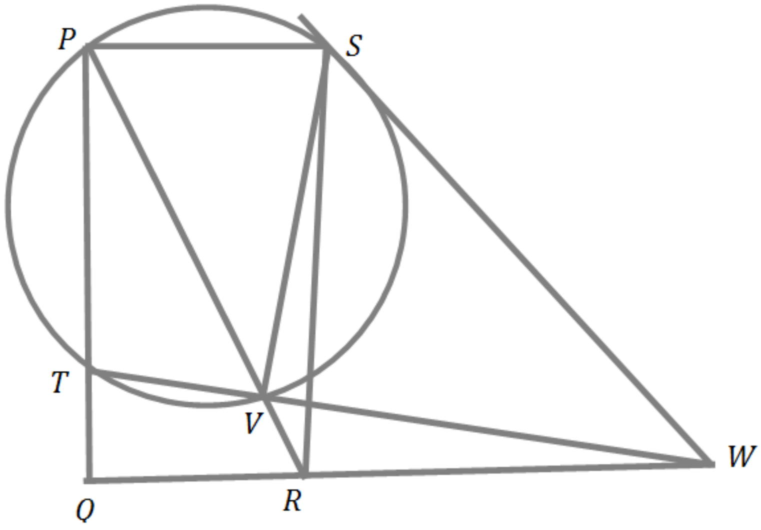
Solution:

I. $Q\hat{P}O = 90°$ [radius ⊥ tangent]
$Q\hat{R}O = 90°$ [radius ⊥ tangent]
∴ PQRO is a cyclic quadrilateral [opp *L*s suppl.]

II. OP = OR (radii)
QP = QR (tans from common pt)
∴ PQRO is a kite [2 pairs of adjacent sides are equal]

III. $V_2 = 90°$ [diag. of a kite intersect at right angles. ∴ OV ⊥ PR]
$T_1 = 90°$
∴ PTOV is a cyclic quadrilateral [ext *L* = opp int. *L*]

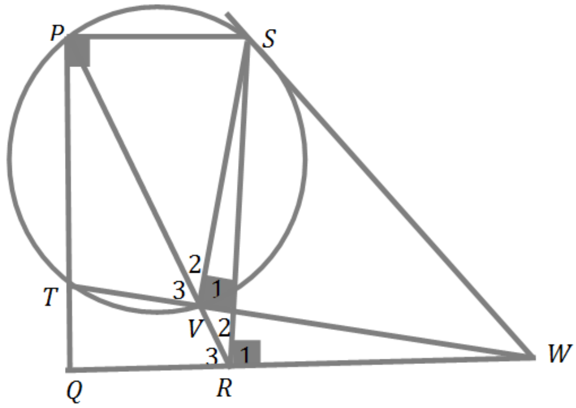
Example 6



PQRS is a rectangle. The circle through S and P cuts PQ at T and PR at V.

Prove:SV $\perp$ TW

I. SVRW is a cyclic quadrilateral



Solution:

I. $\hat{P}$ = 90° [*L* of rectangle]
   V_1 = 90° [ext *L* cyclic quad, SPTV]
   ∴ SV $\perp$ TW

II. $\hat{R}$ = 90° [L^s of rectangle]
   V_1 = 90° [proved in 1]
   ∴ SVRW is a cyclic quadrilateral [SW subtends equal L^s]

Activity 2: Proofs and calculations

Group organisation:	Time:	Resources:	Appendix:
Individual	5 min	• None	None

In your groups you will:

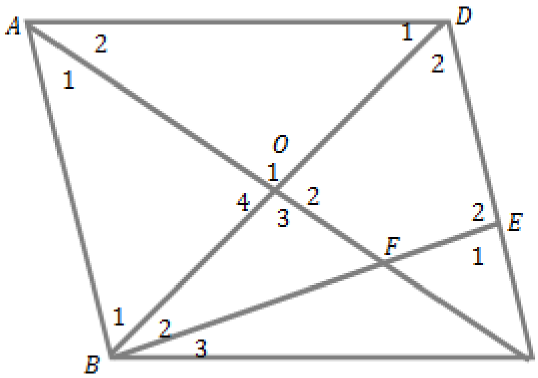
1. Complete the following calculations and write your answers in the space provided.

2. ABCD is a rhombus. The diagonals intersect at O. BE⊥ DC.

Prove:

I. DOFE is a cyclic quadrilateral

II. BCEO is a cyclic quadrilateral



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Conclusion

The theory of quadrilaterals will be integrated into questions in the examination. (See Examination Guidelines for Grade 12, 2014.)

Sub-topic 3: Geometry of Circles

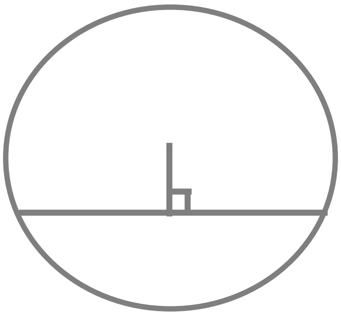
Grade 10	Grade 11	Grade 12
	• Accept results established in earlier Grades as axioms and also that a tangent to a circle is perpendicular to the radius drawn to the point of contact. • Then investigate and prove the theorems of the geometry of circles. • The line drawn from the centre of a circle perpendicular to a chord bisects the chord. • The perpendicular bisector of a chord passes through the centre of the circle. • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre). • Angles subtended by a chord of the circle, on the same side of the chord, are equal. • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length. • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	

Introduction

It is expected from you that you are able to prove the following theorems.

These proofs are examinable.

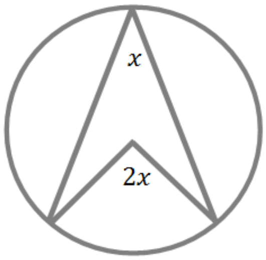
- The line drawn from the centre of a circle perpendicular to a chord bisects the chord. [line from centre ⊥ to chord]



$\Rightarrow$ $\quad AB = BC$

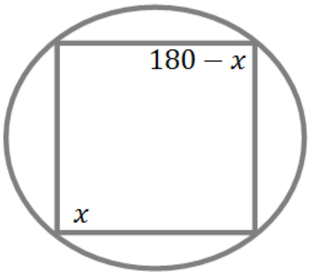
- The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

  [*L* at centre = 2 x *L* at circumference]

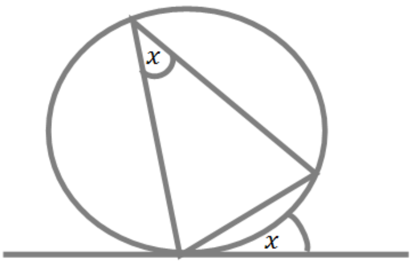


- The opposite angles of a cyclic quadrilateral are supplementary

  [opp L^s of cyclic quad]



- The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment [tan chord theorem]

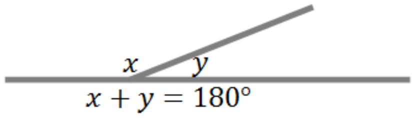


Prior knowledge

Revise the following, from the geometry studied in Grades 8 to 10, as they may be useful in Grade 11 geometry.

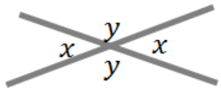
1. The sum of angles on a straight line is 180°

   [L^s on a str line]

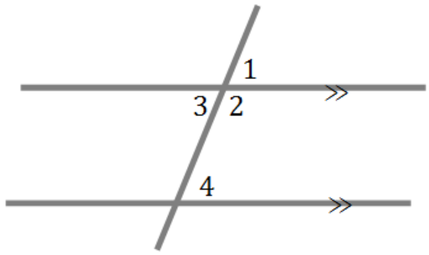


2. Vertically opposite angles are equal

   [vert opp L^s =]



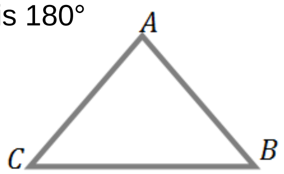
3. If two parallel lines are intersected by a transversal
   - alternate angles are equal (e.g. $\hat{\mathbf{3}} = \hat{\mathbf{4}}$)
   - corresponding angles are equal (e.g. $\hat{\mathbf{1}} = \hat{\mathbf{4}}$)
   - co-interior angles are supplementary ($\hat{\mathbf{2}} + \hat{\mathbf{4}} = 180°$)



4. The sum of the angles of a triangle is 180°

   [L sum in Δ]

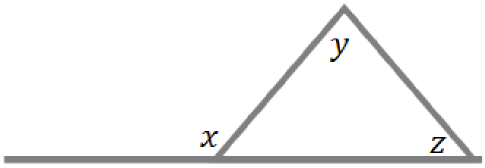
   $\hat{\boldsymbol{A}} + \hat{\boldsymbol{B}} + \hat{\boldsymbol{C}} = 180°$



5. The exterior angle of a triangle is equal to the sum of the interior opposite angles.

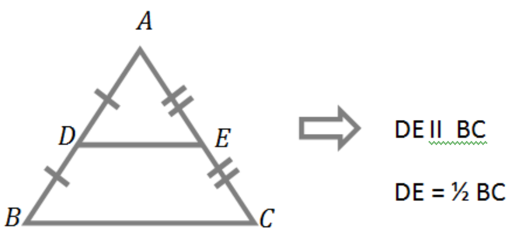
   [ext L of Δ]

   $x = y + z$

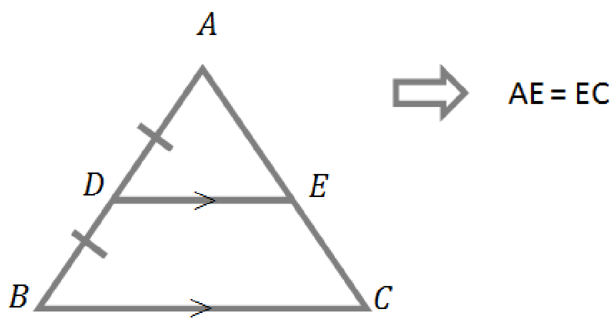


6. Conditions for congruency. [SSS; *S*L*S*; *LLS;* 90°, hyp, side]

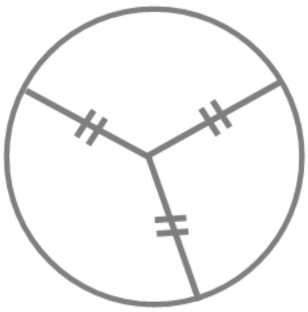
7. <u>Midpoint theorem:</u> If a line joins the midpoints of two sides of a triangle, then the line is parallel to the third side and equal to half the length of the third side. [Midpt Theorem]



8. Intercept Theorem: If a line is drawn through the midpoint of one side of a triangle, parallel to another side, it bisects the third side. [line through midpt II to 2nd side]

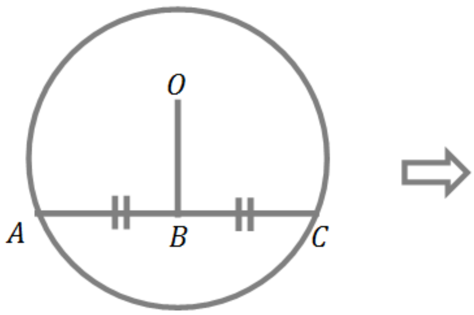


9. Remember radii of a circle are equal.

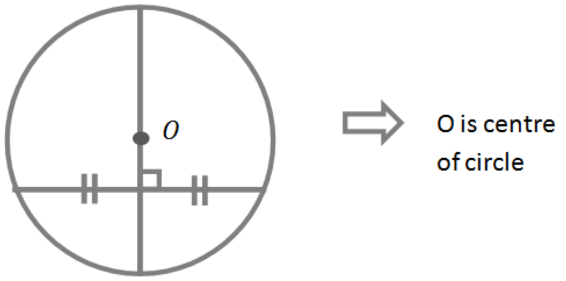


The following is a list of the theorems and their converses as well as an acceptable abbreviation for the reason. (A full list is available in the Mathematics Guidelines 2014 pg. 9, 10 and 11.)

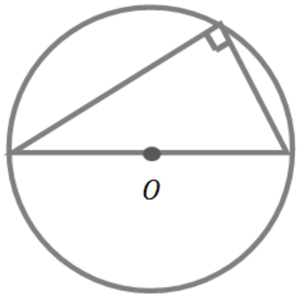
1. A line through the centre of the circle to the midpoint of a chord is perpendicular to the chord. [line from centre to midpt of chord]



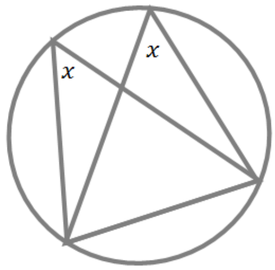
2. The perpendicular bisector of a chord passes through the centre of the circle.
[perp bisector of chord]



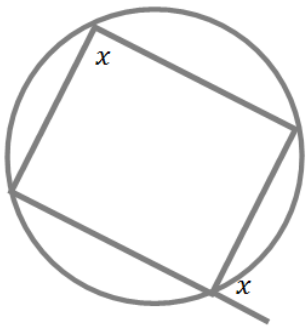
3. The angle subtended by the diameter at the circumference of the circle is 90°.
[$\angle^s$ in the same seg]



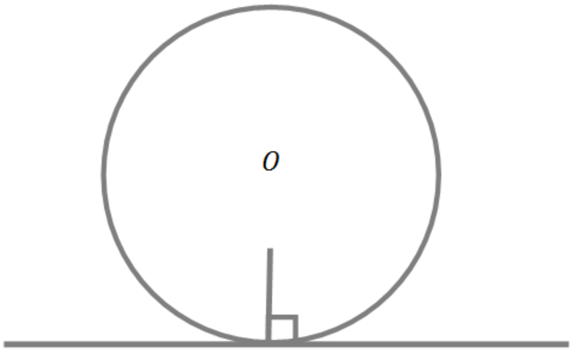
4. Angles subtended by a chord of the circle, on the same side of the chord, are equal.
[$\angle^s$ in the same seg]



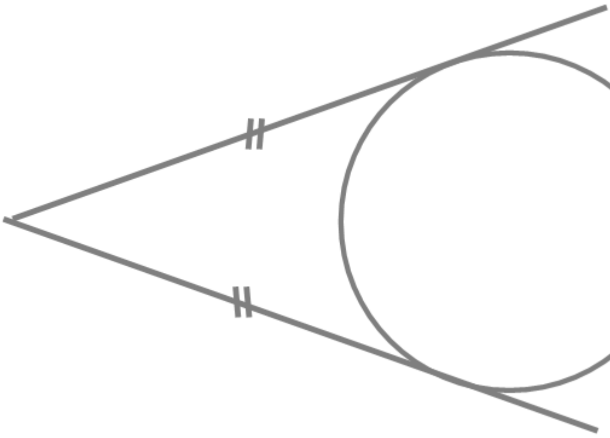
5. The exterior angle of a cyclic quadrilateral equals the opposite interior angle.
[ext $\angle$ of cyclic quad]



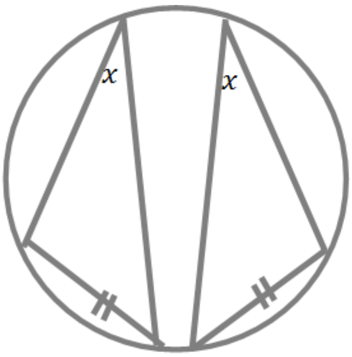
6. A tangent to a circle is perpendicular to the radius (or diameter) which is drawn to the point on contact.
[tan ⊥ radius]



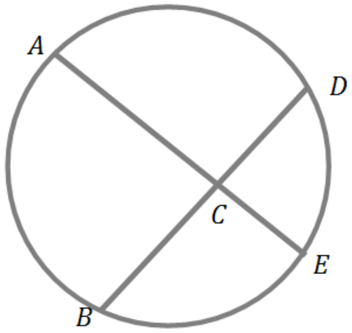
7. Two tangents drawn to a circle from the same point outside a circle are equal in length.
[tans from same pt]



8. Equal chords subtend equal angles at the circumference of the circle.
[equal chords; equal L^s]

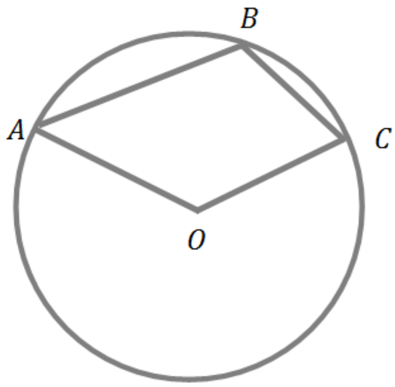


Misconceptions



In the diagram AB does NOT subtend $A\hat{E}B$.

C is NOT the centre of the circle and C is NOT on the circumference of the circle.



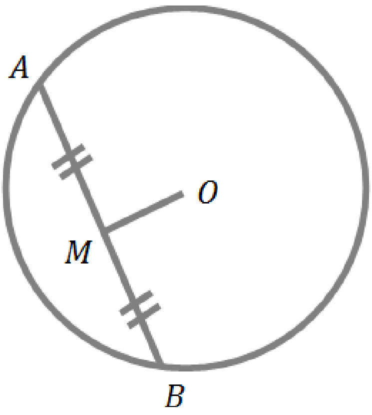
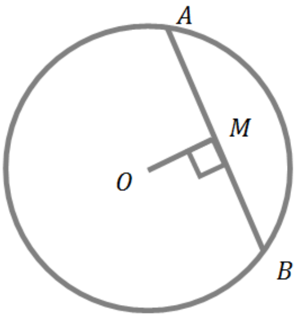
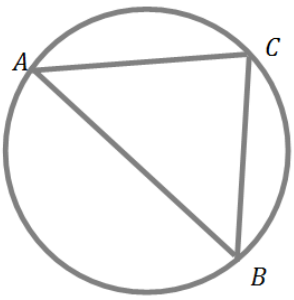
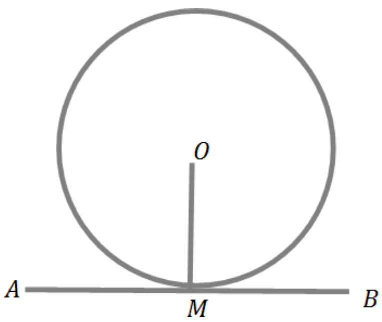
In the given diagram ABCO is NOT a cyclic quadrilateral. Point O is not on the circumference of the circle and OA and OC are not chords.

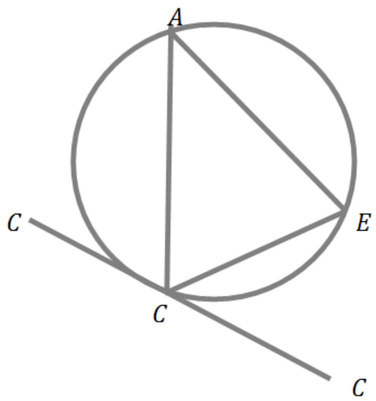
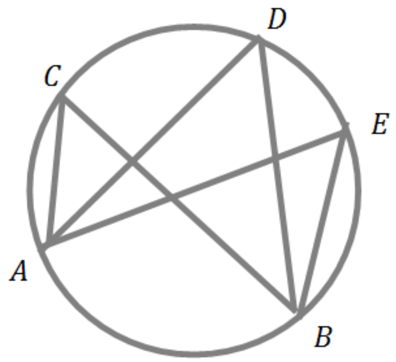
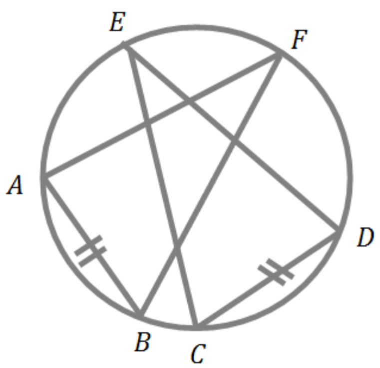
Activity 1: Investigating Circles

Group organisation:	Time:	Resources:	Appendix:
Pairs	1 hour	• Numerous sheets of A4 paper • Mathematics sets	None

In your groups you will:

1. Use the correct instruments to draw the diagrams and measure the angles as instructed.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions could be.
3. Every group will have an opportunity to provide feedback.

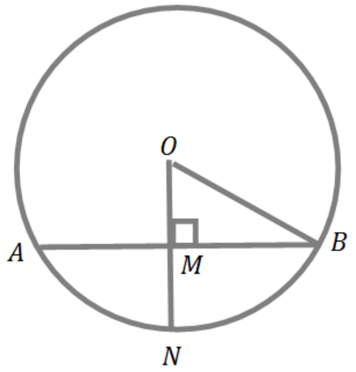
	Draw a nice big circle and mark the centre, O. Draw any chord, AB, and mark the midpoint of the chord, M. Draw a line from the centre, O, to the midpoint of the chord, M. Measure: $O\hat{M}B$ = ____________ What can you deduce?
	Draw a big circle. Draw any chord, AB. Draw a line from the centre of the circle perpendicular to the chord. Measure: AM = ________ and MB = ________ What can you deduce?
	Draw a big circle with a diameter, AB. Draw inscribed angle, $\hat{C}$, subtended by AB. Measure: $\hat{C}$ = ____________ What can you deduce?
	Draw a nice big circle. Draw a tangent, AB, to the circle. Draw a line from the centre of the circle to the point of contact, M. Measure: $O\hat{M}B$ = ____________ What can you deduce?

	Draw a big circle. Draw a tangent, BD, to the circle. C is the point of contact. Draw a chord AC that subtends $\widehat{E}$ in the big segment. Measure: $A\hat{C}B$ = _______ and $\widehat{E}$ = _______ What do you notice?
	Draw a circle. Let arc, AB, subtend a few inscribed angles $\hat{C}$, $\widehat{D}$ and $\widehat{E}$. Measure: $\hat{C}$ = __________ $\widehat{D}$ = ________ and $\widehat{E}$ = ________ What do you notice?
	Draw a big circle. Draw two chords, AB and CD, such that AB = CD. AB subtends inscribed $\widehat{F}$ and CD subtends inscribed $\widehat{E}$. $\hat{C}$, $\widehat{D}$ and $\widehat{E}$. Measure: $\widehat{E}$ = __________ $\widehat{F}$ = ________ What do you notice?

Worked Examples 1-3

The following examples will further help you understand this concept and explain this easily to your learners. Please make notes as the facilitator is talking and don't forget to ask as many questions as possible so as to clarify any misconceptions that may occur.

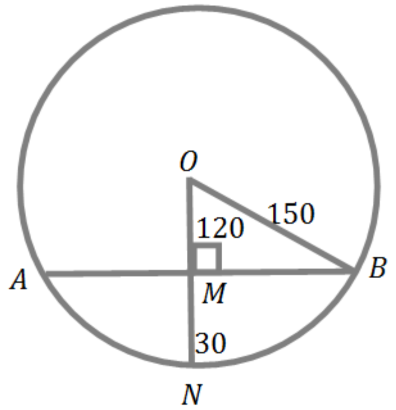
Example 1



AB is a chord of the circle with the centre, O. ON ⊥ AB and intersects AB in M.
N is on the circumference of the circle.

If MN = 30 mm and the radius of the circle is 150mm, calculate the length of AB.

Solution:

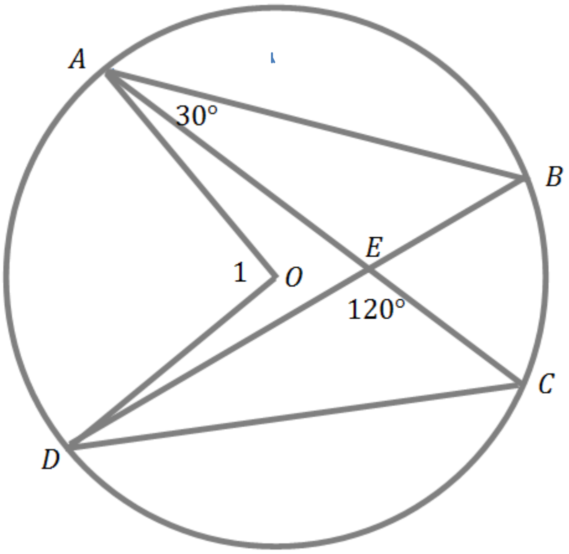


ON = 150 (radius)
MN = 30
∴ OM = 150 – 30
= 120mm ✓

$MB^2 = OB^2 - OM^2$ ✓ (Pythagoras)
$= 150^2 - 120^2$
=22500 – 14400
= 8100
∴ MB = $\sqrt{8100}$
= 90 mm ✓

AB = 2 x MB [line from centre ⊥ to chord bisects the chord]
= 2 x 90
= 180mm ✓✓

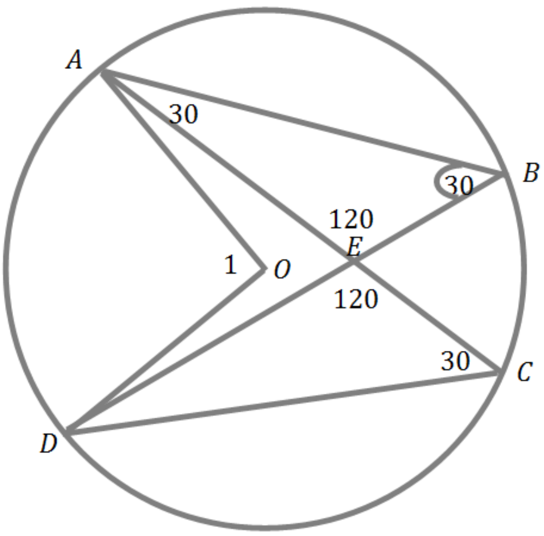
Example 2



O is the centre of the circle. $D\hat{E}C = 120°$ and $C\hat{A}B = 30°$

Calculate, giving reasons, the magnitude of:

1. $\hat{O}_1$
2. $\hat{C}$

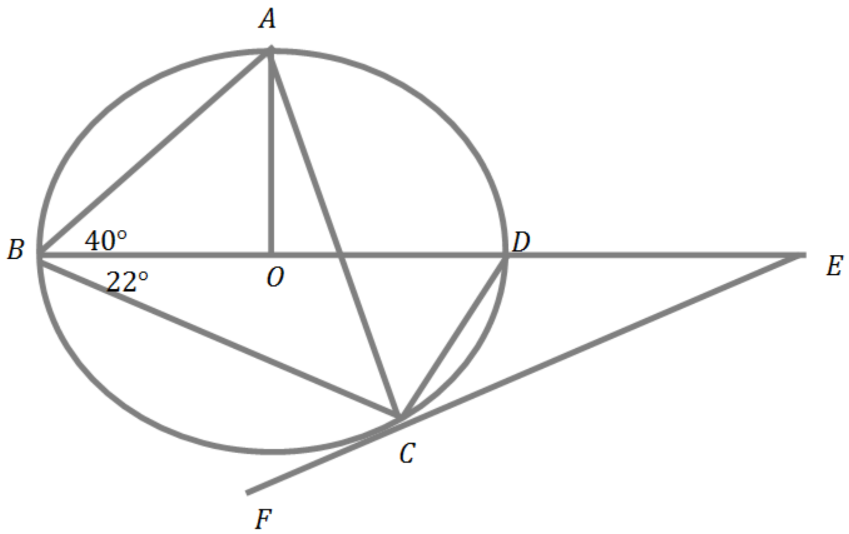


Solution:

1. $A\hat{E}B = 120°$ [vert. opp $\angle^s$] ✓
   $\hat{B} + 30 + 120 = 180$ [sum $\angle^s$ Δ AEB = 180°]
   $\hat{B} = 30°$
   $\therefore \hat{Q}_1 = 2 \times \hat{B}$ [$\angle$ at centre =] ✓
   $= 2 \times 30$ [2 x $\angle$ at circumference]
   $= 60°$ ✓

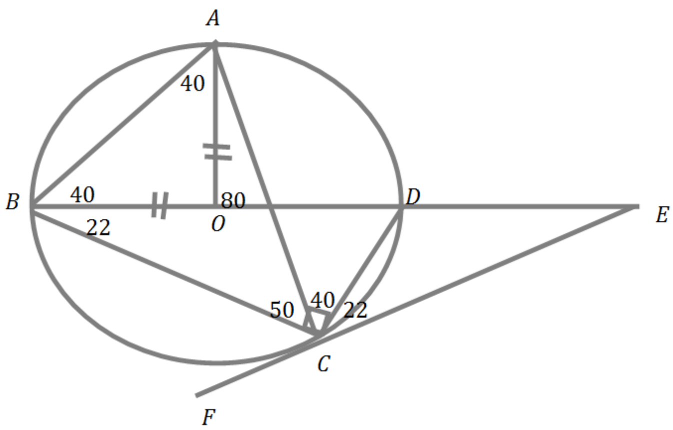
2. $\hat{C} = \hat{B}$ [$\angle^s$ in the same segment.]
   $= 30°$ ✓

Example 3



FCE is a tangent. BD is a diameter. O is the centre of the circle.

$A\hat{B}D = 40°$ and $D\hat{B}C = 22°$



Calculate, giving reasons

I. $B\hat{A}O$
II. $A\hat{O}D$
III. $A\hat{C}D$
IV. $A\hat{C}B$
V. $A\hat{C}E$

Solution:

I. $B\hat{A}O = 40°$ [Δ BOA isosceles, OB = OA radii] ✓

II. $A\hat{O}D = 80°$ [ext L of a Δ = sum interior opposite L^s] ✓

III. $A\hat{C}D = A\hat{B}D$ [subt are AD] ✓
$= 40°$

IV. $A\hat{C}B = 90 - 40$
$= 50°$ [$B\hat{C}D = 90°$, L in semi ⊙] ✓

V. $D\hat{C}E = 22°$ [angle between chord CD and F tan FCE = L CBD in opposite segment]
∴ $A\hat{C}E = 22 + 40$
$= 62°$

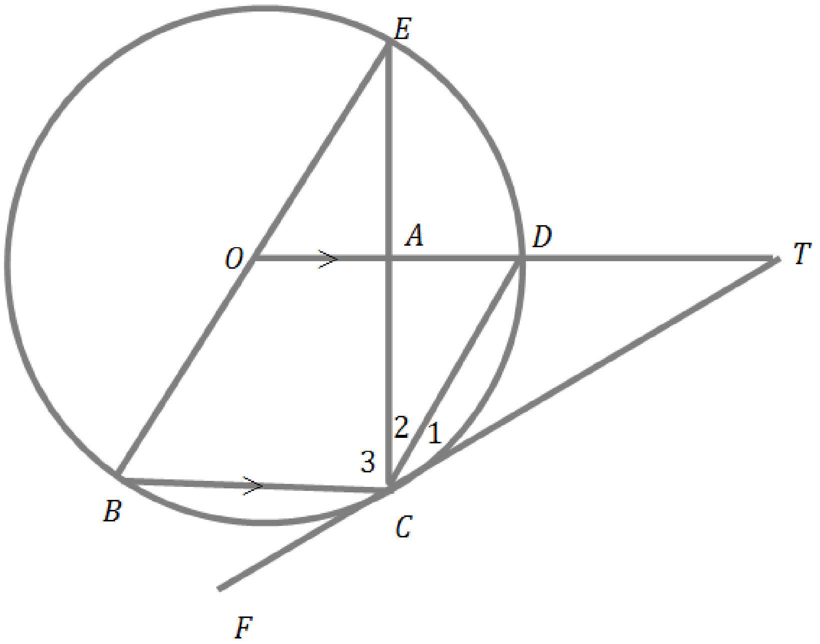
Activity 2-4: Proofs and calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	1 hour	• Flipchart paper • Whiteboard markers	None

In your groups you will:

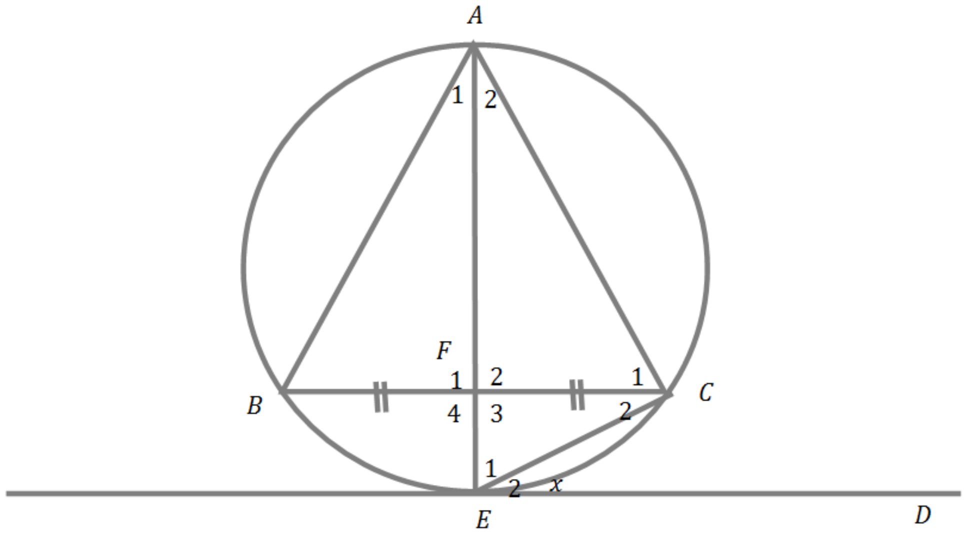
1. Select a scribe and a spokesperson for this activity only.
2. Use the flipchart and permanent markers to answer the questions that follow. Be sure to explain to the groups during your report back how you would teach these concepts and what the possible misconceptions could be.
3. Every group will have an opportunity to provide feedback.

Activity 2



O is the centre of the circle. BE is a diameter.
FT is a tangent to the circle at C. OD II BC.

I. Write down the size of $A\hat{C}B$. Give a reason.
II. Prove EA = AC.
III. If $E\hat{O}A = 55°$. Calculate, giving reasons, the size of $E\hat{C}T$.

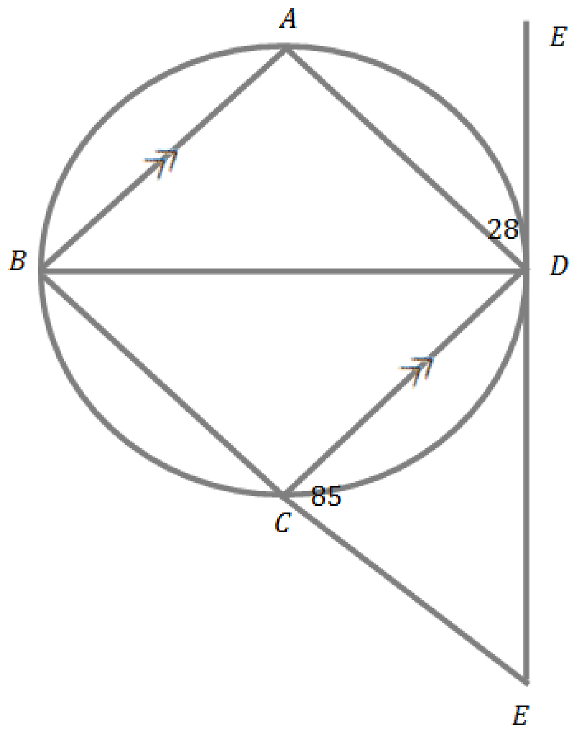


O is the centre of the circle. BF = FC. DE is a tangent to the circle at E.
$C\hat{E}D = x$

Determine, giving reasons, the sizes of the following angles in term of x

I. $\hat{A}_2$
II. $\hat{E}_1$
III. $\hat{C}_2$
IV. $\hat{A}_1$

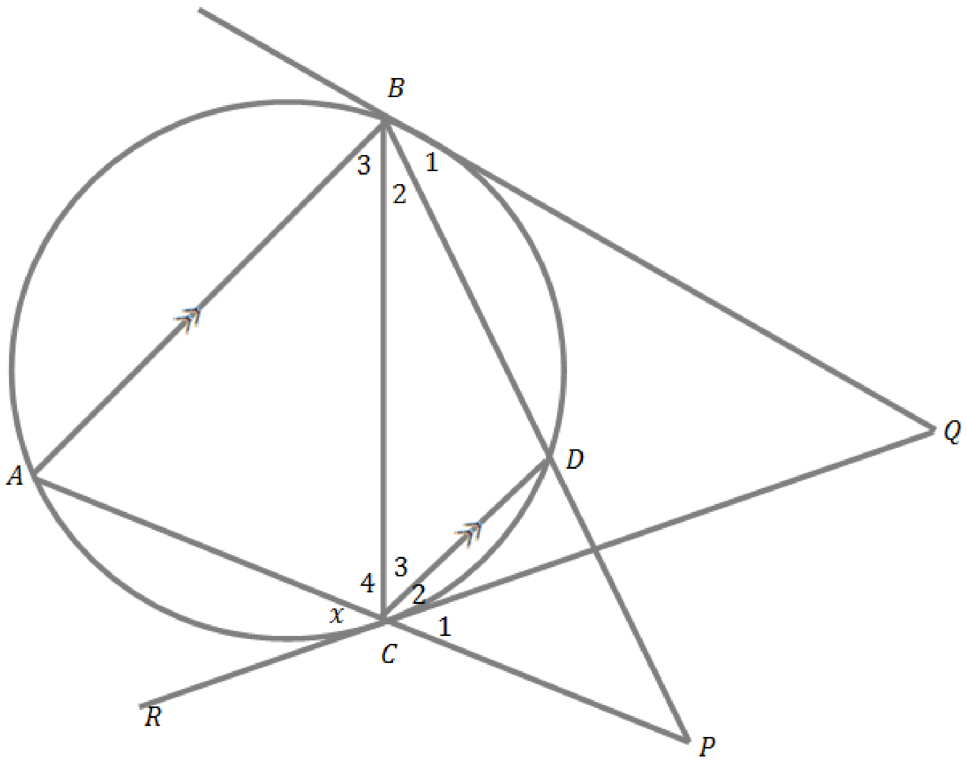
Activity 4



ABCD is a cyclic quadrilateral. AB II DC. FDE is a tangent.
$D\hat{C}E = 85°$ and $A\hat{D}F = 28°$

Calculate, giving reasons, the sizes of:

I. $A\hat{B}D$
II. $B\hat{D}C$
III. $B\hat{A}D$
IV. $D\hat{B}C$



QB and QC are tangents to the circle at B and C respectively.
BA II DC Let $A\hat{C}R = x$

Name four angles equal to x. Give reasons.

Conclusion

In Euclidean Geometry diagrams (figures) are NOT drawn according to a scale. E.g. If it is not given that lines are equal in length, that it is the centre of a circle or that it is a tangent or a diameter, etc. then it is not, unless you are asked to prove it to be so.

Corollaries derived from the theorems and axioms are necessary in solving geometry problems and their converses to solve riders.

Tip: Learn the theorems by writing them down in pictorial form.

Sub-topic 4: Integrated Examples

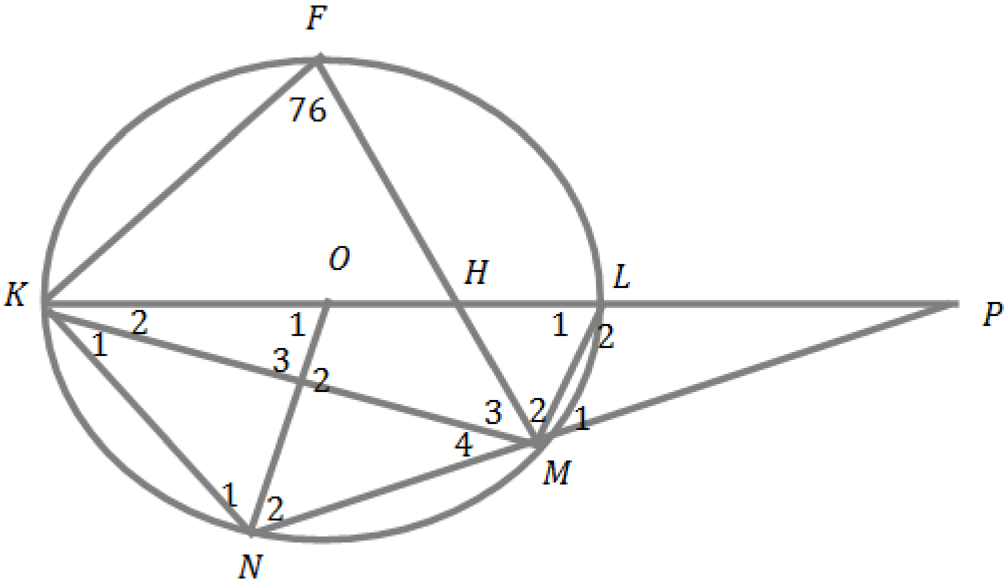
Activity 1: Proofs and calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	1 hour	• A4 paper • Thin felt pens • Flipchart paper • Whiteboard markers • Glue sticks	None

In your groups you will:

1. Use the A4 pages to perform the calculation or prove the rider allocated to you.
2. Once you have completed this, use the glue and stick the A4 page/s on the flipchart. Now proceed and write down on the same flipchart an interesting way that you would teach this to your learners. Also indicate how you would address misconceptions.
3. Stick your completed chart on the wall.
4. We will conduct a gallery walk and every group will have an opportunity to discuss the contents of their charts.

Question1



Calculate, giving reasons, the sizes of:

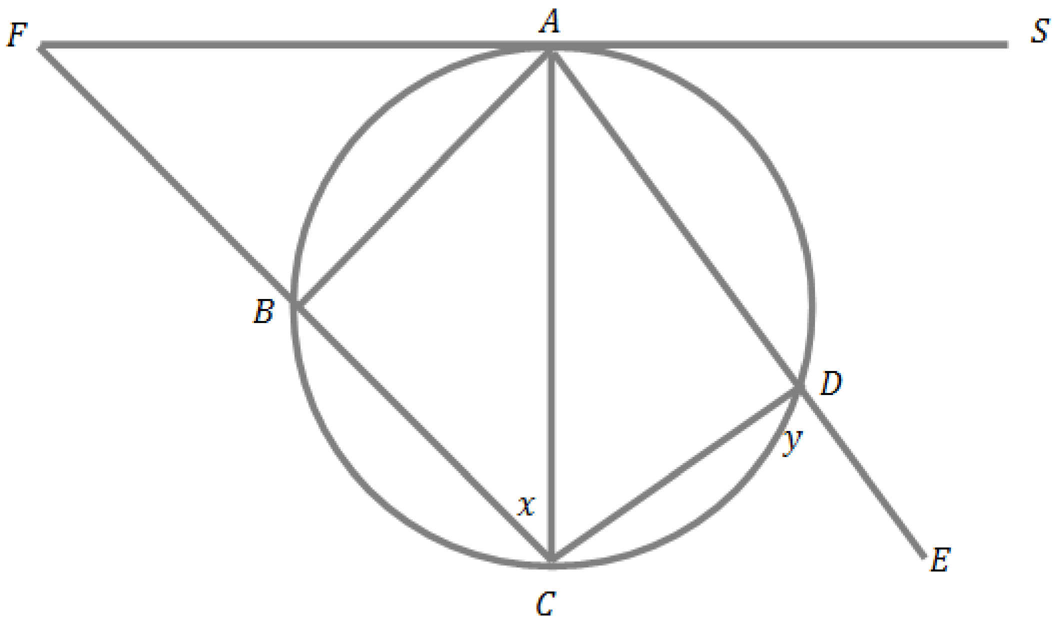
I. $\hat{L}_1$

II. $\hat{O}_1$

III. $\hat{M}_4$

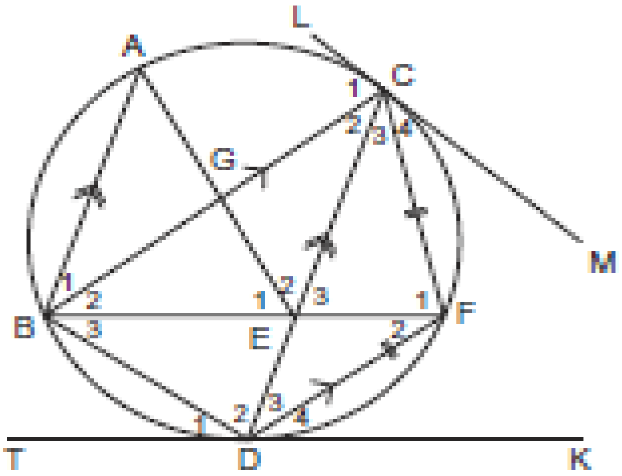
IV. $\hat{N}_1 + \hat{N}_2$
V. $\hat{M}_1$
VI. Prove That KG = GM

Question 2



I. Name one angle equal to x. Give a reason.
II. Name two angles equal to y. Give reasons.
III. Express $\hat{F}$ in terms of x and y.

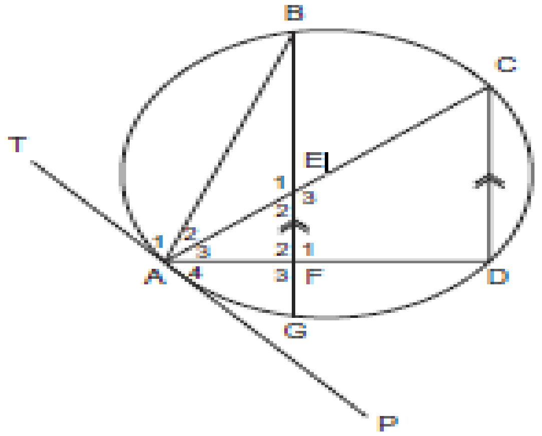
Question 3



I. Circle ABDFC has tangents TDK and LCM at D and C respectively.
II. CDIIAB and BCIIDF.
III. CF = FD.

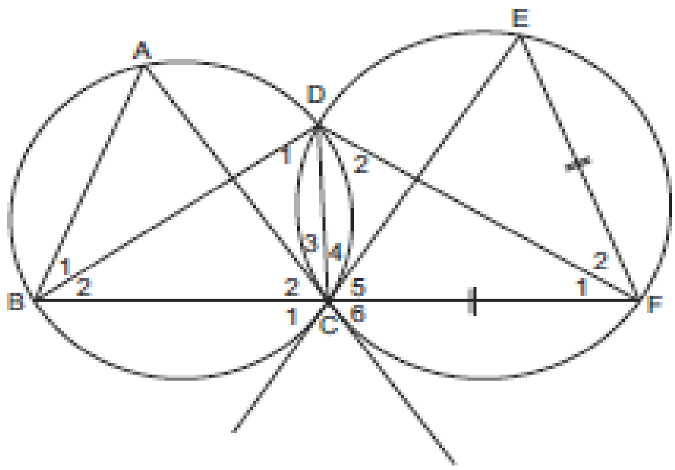
IV. Lines BF and CD intersect at E and lines AE and BC intersect at G.
V. Prove that EB = EC.
VI. Prove that BF = DC.

Question 4



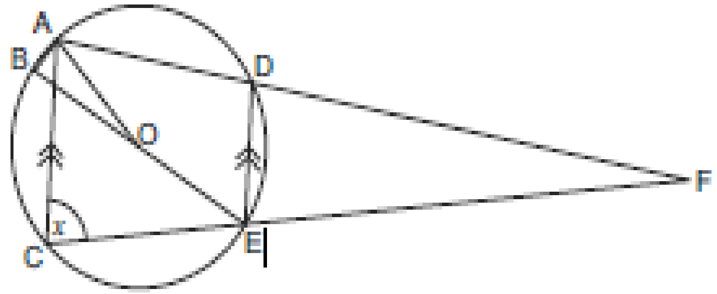
Circle ABCD has a tangent at A. BGIICD and AD intersects BG at E and F respectively. Prove that TAP is a tangent to the circle through A, E and F.

Question 5



In the sketch, circles ABCD and CDEF intersect at D and C. AC is a tangent to circle CDEF at C, and EC is a tangent to circle ABCD at C. EF = CF and $\hat{E} = x$.
Prove that:
I. CD bisects $B\hat{D}F$
II. ΔABC is an isosceles triangle

Question 6



In the sketch, the circle with centre O has BE as a diameter and points A, B, C, D and E on the circle. CE and AD are produced to F and ACIIDE. Show all working and give reasons for your answers. If $\hat{C} = x$:

I. Find five other angles equal to *x.*

II. Prove that $A\hat{O}B = \hat{F}$

III. Prove that FA = FC.

References

Eadie, A. & Lampe, G. (2013). Gr 12 Wiskunde Vraestel 3. Die Antwoord reeks. Die Antwoord: Claremont, Kaapstad

Macmillan. (2014). Learn Xtra Live Grade 10 + 12 Mathematics Study Aid.

Mathematics Grade 10 Papers with Solutions. Examenhulp, Lynwoodrif

Rapport 7 September 2014

Van Kradenburg, E.,, Vermeulen, N., Prins, E. & Otto, H. (2010). Pythagoras 10. Fisichem Matieland.

Van Kradenburg, E., Vermeulen, N. & Prins, E.. Pythagoras 11. Fisichem Matieland

Van Kradenburg, E., Vermeulen, N. & Prins, E. (2014).Pythagoras 12. Fisichem Matieland.