

Module 1: Functions

Sub-topic 1: Introduction to functions

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
The concept of a function Relationships between variables using words, tables, graphs, and formulae Domain and range		Definition of a function

Introduction

When introducing functions to learners in grade 10, it is important to refer to the mathematics which they have come across in their earlier grades. Some examples of these are:

- Working with the different operations
- Checking for relationships (between sets of numbers, in formulae, etc.)
- Substitution in equations/formulae
- Sketching, finding the equation of and interpreting the graph of $y = mx + c$ (a straight line)

The approach used in grade 10 is developmental. In grade 12, learners are introduced to a more formal approach to a function. In this module, we used both approaches, starting with a developmental approach, which makes use of learners' prior knowledge.

 Activity 1: Introduction to functions			
Group organisation:	Time:	Resources:	Appendix:
Groups of 6	10 min	• Flip chart • Permanent markers.	None
In your groups you will: 1. Select a scribe and a spokesperson for this activity only. (Rotate from activity to activity) 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

1. Complete the table below:

x	0	1	2	3	4
x^2					

- 1.1 Write down some ordered pairs for the above table.
- 1.2 Write down the equation represented by the information in the table.
- 1.3 Draw the graph.
- 1.4 Is there any restriction on the domain and range? Why?
- 1.5 Write down the domain and range.

2. The number of diagonals in a polygon is given by the following table:

Number of sides (x)	3	4	5	6	7
Number of diagonals (y)	0	2	5		

- 2.1 Complete the table.
- 2.2 Draw the graph.
- 2.3 Is there any restriction on the domain and range?
- 2.4 Write down the domain and range.
- 2.5 Determine the equation showing y in terms of x .

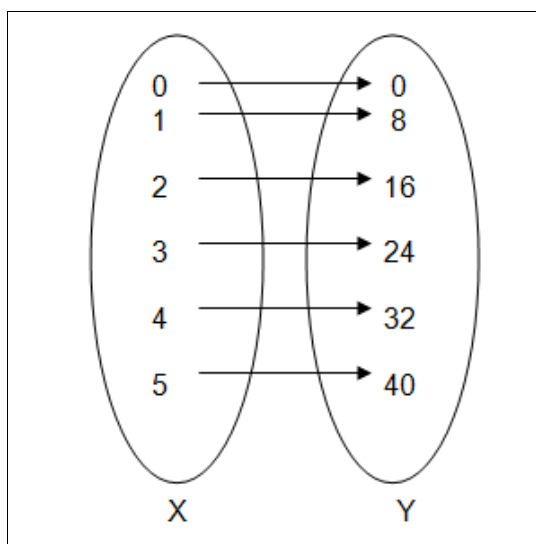
Consolidation & Terminology

The situations/relationships described in activity 1 represent functions. **Use this information to explain to your learners in very simple terms what a function is. Even though this is handled in detail and formally assessed in grade 12, for grade 10 learners this is intuitive.**

NB: On page 131 of the Report on the 2013 National Senior Certificate (Diagnostic report), it was noted that learners struggled to identify or define a function. They were not clear about terminology such as “one-to-many” and “many-to-many”.

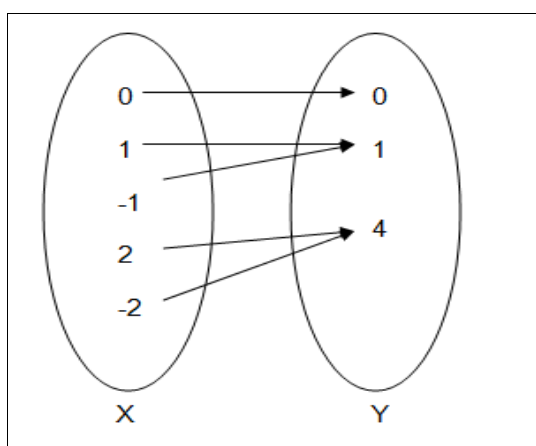
Definition:

A function is a rule by means of which each element of a first set, called the domain, is associated with **only one element** of a second set, called the range. Each element of the range is an image of corresponding elements of the range. Look at the following Venn diagrams.



Not all elements of the domain and range are shown in the diagrams. We see that there is a one-to-one correspondence between the elements of the domain and elements of the range.

If we represent number 1 from exercise 1 in a Venn diagram we get:



We see that:

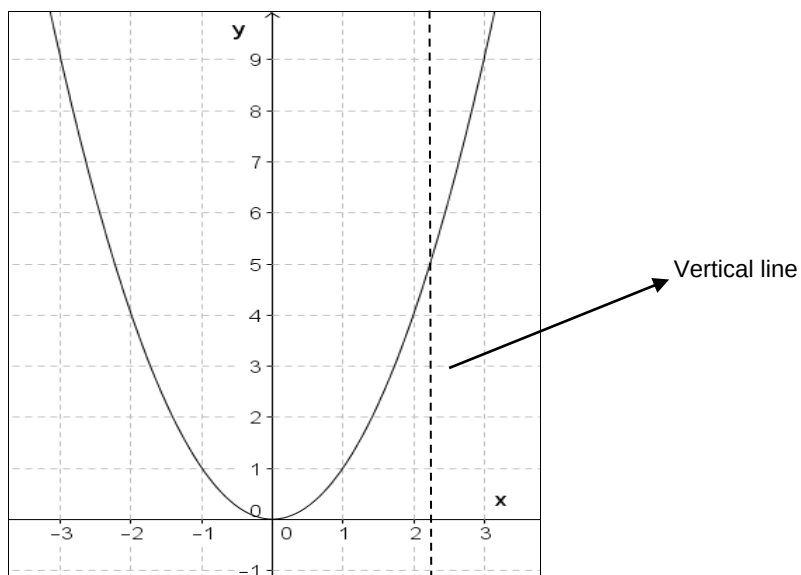
- Both 1 and -1 in the domain correspond to 1 in the range.
- Both 2 and -2 in the domain correspond to 4 in the range.

This correspondence is many-to-one. When the correspondence is **one-to-one** or **many-to-one**, we have functions; if the correspondence is **one-to-many** or **many-to-many** we do not have functions.

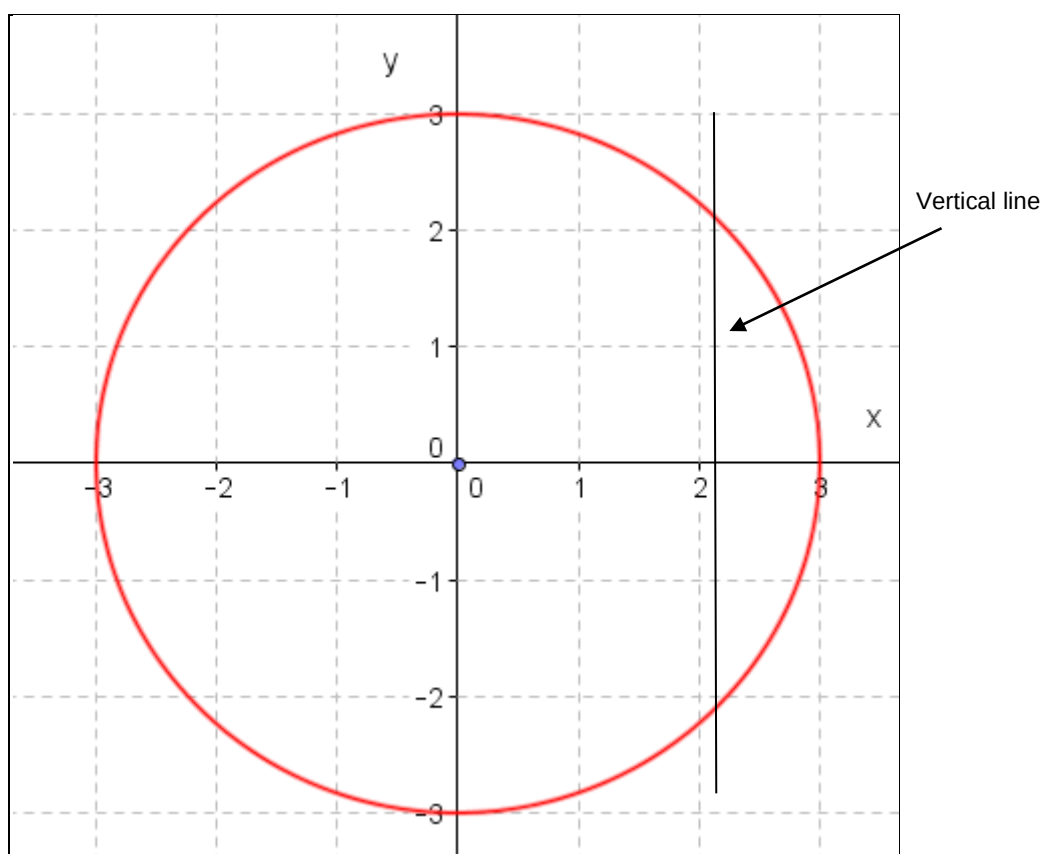
When determining whether a graph represents a function or not we make use of the “**correspondence**” reasons.

However, we could also use a simple test called the “vertical line” test to determine whether a graph represents a function or not.

If the vertical line cuts the graph at only one point, the graph represents a function; if it cuts at more than one point then the graph does not represent a function:



The vertical line cuts the graph at only one point so the graph will represent a function.



The vertical line cuts the graph at two points so the graph will not represent a function.

(The circle and semi-circle is not prescribed in CAPS for this section. It is merely used to show the vertical line test.)



Worked Examples 1 & 2: Functional Notation (10 mins)

The facilitator will now explain these suitable examples of functional notation.

1. Remember to make notes as the facilitator is talking
2. Ask as many questions as possible so as to clarify any misconceptions that may occur.

1. Given that $f(x) = x^2 - 4$

Determine the following:

1.1 $f(1)$

1.2 $f(-3)$

1.3 The value(s) of x if $f(x) = 0$

Solution

Once again, for 1.1 and 1.2 it is simple substitution:

1.1

$$\begin{aligned} f(1) &= (1)^2 - 4 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

1.2

$$\begin{aligned} f(-3) &= (-3)^2 - 4 \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

- 1.3 For this question please note that $f(x) = 0$ is not the same as $f(0)$.

You may show learners this difference on a graph.

$$\begin{aligned} f(x) = 0 &\Rightarrow x^2 - 4 = 0 \\ &\Rightarrow (x + 2)(x - 2) = 0 \\ &\Rightarrow x = -2 \text{ or } x = 2 \end{aligned}$$

We note that in this question we used factorisation to obtain the values of x .

1. Given the following function:

$$f(x) = \sqrt{x^2 - 9}$$

Determine:

2.1 $f(5)$

2.2 The value(s) of x if $f(x) = 2$.

2.3 The domain of f .

2.4 The range of f.

Solution

$$2.1 \ f(5) = \sqrt{(5)^2 - 9} = \sqrt{25 - 9} = \sqrt{16} = 4$$

2.2

$$f(x) = 2$$

$$\Rightarrow \sqrt{x^2 - 9} = 2$$

$$\Rightarrow x^2 - 9 = 4 \quad (\text{square both sides})$$

$$\Rightarrow x^2 = 13$$

$$\Rightarrow x = \pm\sqrt{13}$$

2.3 The domain of f exists when $x^2 - 9 \geq 0$ (why? – this is a restriction)

$$x^2 - 9 \geq 0 \quad (\text{this is a simple quadratic inequality})$$

$$\Rightarrow (x + 3)(x - 3) \geq 0$$

$$\Rightarrow x \leq -3 \text{ or } x \geq 3$$

NB: You may have to briefly revise how to solve quadratic inequalities.

Thus, the domain is: $\{x: x \leq -3, x \in \mathbb{R}\} \cup \{x: x \geq 3, x \in \mathbb{R}\}$

2.4 Note that $\sqrt{\quad}$ indicates positive square root. Thus, our range can never be negative.

$$\text{Range} = \{y: y \geq 0, y \in \mathbb{R}\}$$



Activity 2, 3 & 4: Functional Notation

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	10 min	- Flip chart - Permanent markers.	None
In your groups you will: - Select a scribe and a spokesperson for this activity only. (Rotate from activity to activity) - Use the flipchart and permanent markers and answer the questions as per the activity.			

2. Use $f(x) = 2x - 1$ and $g(x) = -x^2 + 4$ to determine:

$$2.1 \ f(-3)$$

$$2.2 \ \frac{1}{2}g(2) + 3$$

3. Use $f(x) = -2x + 1$; $g(x) = \frac{x^2}{x-3}$ and $h(x) = \sqrt{5-x}$ to determine:

3.1 $h(-4) - f(-1)$

3.2 x if $g(x) = 0$

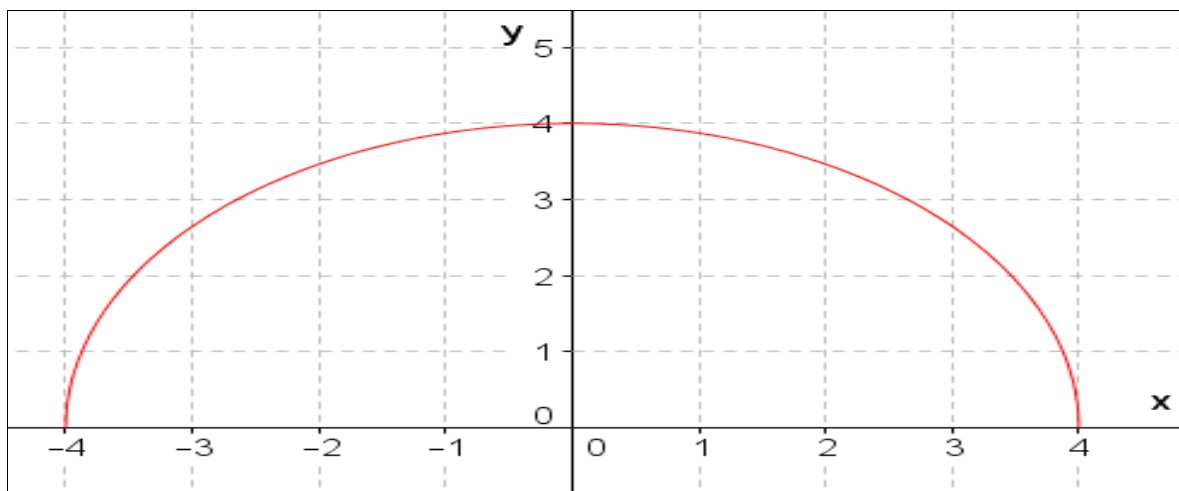
3.3 the domain of g

4. Use $g(x) = \frac{\sqrt{x+2}}{x-3}$ to determine the domain of g .

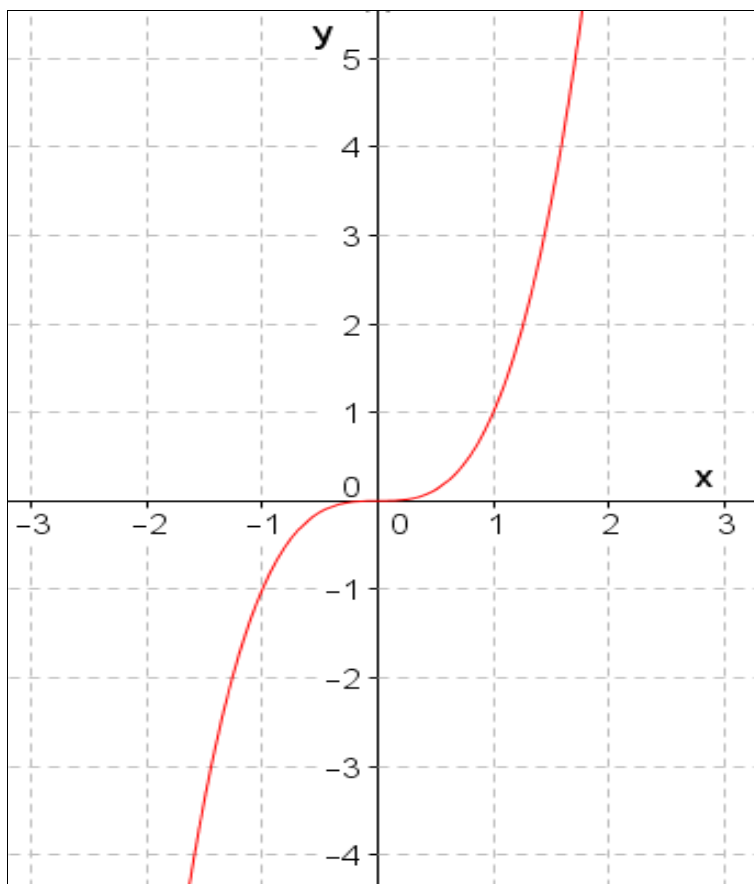
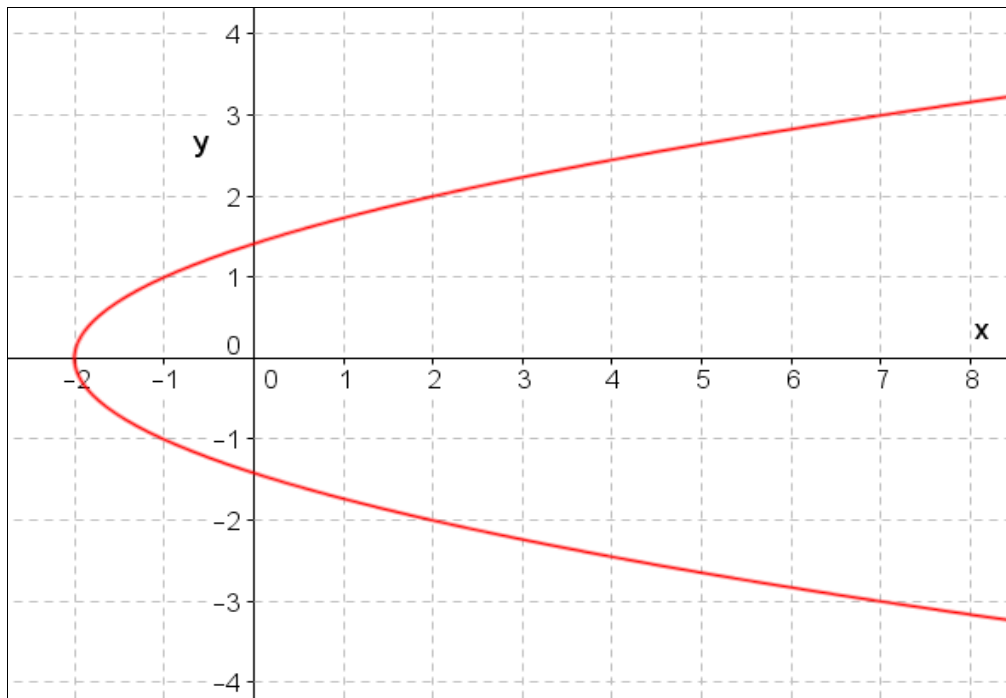


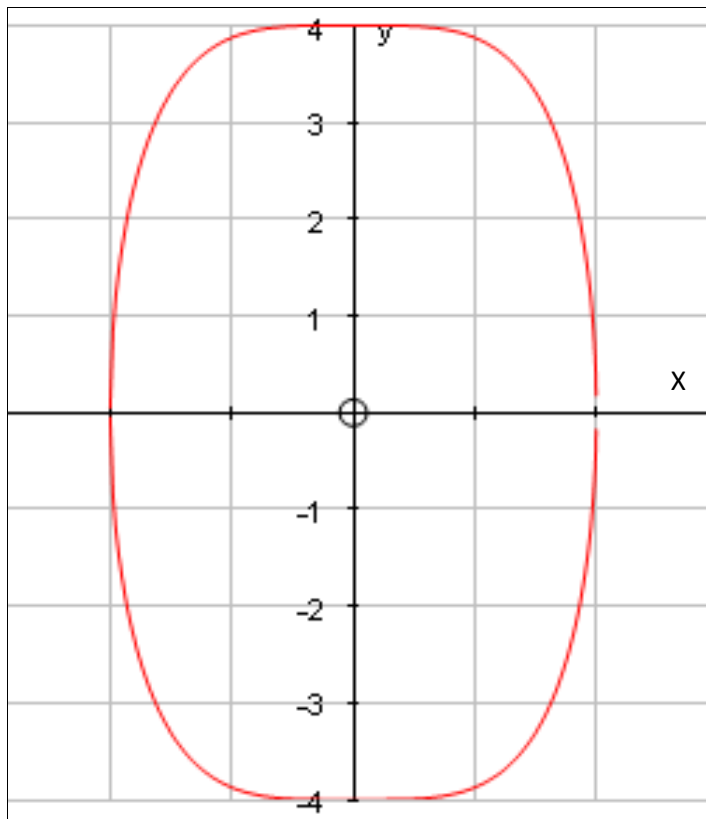
Activity 5: Types of Correspondence

Group organisation:	Time:	Resources:	Appendix:
Individual	5 min	Participants Handout	None
1. Study the graphs that follow and state whether they represent functions or not. Give your reason/s in terms of correspondence. 2. Write your answers in the space provided.			



(The circle and semi-circle is not prescribed in CAPS for this section. It is merely used to show the vertical line test.)





Enrichment

Given $f(x) = \frac{\sqrt{x^2 + 2}}{x + 3}$

We note that f represents a function because for each x -value (of the domain), there is only one y -value (of the range).

To determine the correspondence, we try $x = -1$ and $x = 1$.

$$f(-1) = \frac{\sqrt{x^2 + 2}}{x + 3} = \frac{\sqrt{(-1)^2 + 2}}{-1 + 3} = \frac{\sqrt{3}}{2}$$

$$f(1) = \frac{\sqrt{x^2 + 2}}{x + 3} = \frac{\sqrt{(1)^2 + 2}}{1 + 3} = \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

Thus, the correspondence is one-to-one.

Misconceptions

Teachers should pay careful attention when working with functions which have restrictions as these affect the domain and range of the said function. Learners tend to ignore restrictions on certain functions.

Examples

1. $g(x) = \sqrt{x-3}$ is a restricted function as $x-3$ lies under the square root so $x-3 \geq 0$, that is $x \geq 3$
2. $h(x) = \frac{1}{x+2}$ is also a restricted function as $x+2$ cannot be 0 so $x \neq -2$.
3. The domain of $f(x) = 2x^2 - 3x + 1$ is not restricted but its range is restricted.

Please make learners aware of the following:

- The difference between $g(0)$ and $g(x)=0$
- The difference between $g(2)$ and $g(x)=2$

Conclusion

When working with functions it is important to explain to learners how to get the correspondence. They should be able to look at any table, set of ordered pairs, equation or graph and state what the correspondence is. Functions are a major portion of the Mathematics paper 1 curriculum.

NB: In each lesson please ensure that the lesson incorporates the following:

- Kinaesthetic learning – learners are given hands-on activities/exercises to work through.
- Auditory learning – using one's voice effectively without confusing learners.
- Visual learning – learners must be able to see. In the cases of graphs, there should be some colour and the various features shown.

Sub-topic 2: Introduction to quadratic functions

$$\{(x;y): y = a(x+p)^2 + q\}$$

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
Point-by point plotting of $y = x^2$ Shape of functions, domain & range; axis of symmetry; turning points, intercepts on the axes The effect of a and q on functions defined by $y = a.f(x) + q$ where $f(x) = x^2$ Sketch graphs and find the equation of graphs and interpret graphs	Revise the effect of the parameters a and q and investigate the effect of p and the function defined by $y = a(x+p)^2 + q$ Sketch graphs and find the equation of graphs and interpret graphs	

Introduction

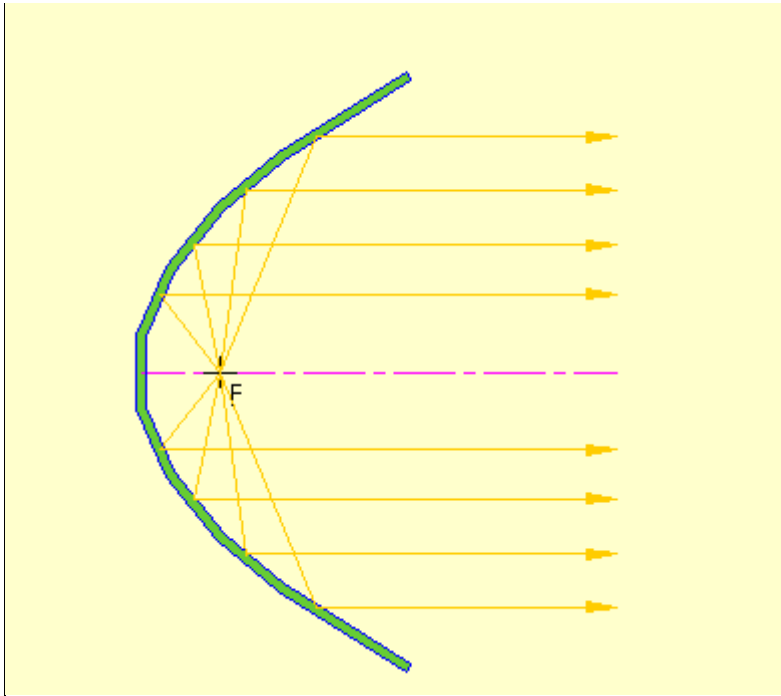
It is important to note that the quadratic function is also known as the parabola. Study the picture below:



This is an example of a parabola in real life. It is the St Louis Arch, which is located in the state of Missouri in the United States of America.

Accessed from: <http://www.tattoodonkey.com/parabolas-in-real-life-tattoo/2/>

Another example of a parabola in real-life is to be found in car headlights.



Accessed from <http://www3.ul.ie/~rynet/swconics/UP.htm>

The general quadratic function

The general formula for the quadratic function is: $y = a(x + p)^2 + q$, where a , p and q are all real numbers. However $a \neq 0$. Why?

We can simplify the quadratic function as follows:

$$\begin{aligned}y &= a(x + p)^2 + q \\&= a(x^2 + 2px + p^2) + q \\&= ax^2 + (2ap)x + (ap^2 + q) \\&= ax^2 + bx + c \text{ (where } b = 2ap \text{ and } c = ap^2 + q)\end{aligned}$$

$y = ax^2 + bx + c$ is another way of writing the general formula of a quadratic function.

Quadratic functions in the form $y = a(x + p)^2 + q$ or $y = ax^2 + bx + c$ represent the parabola.

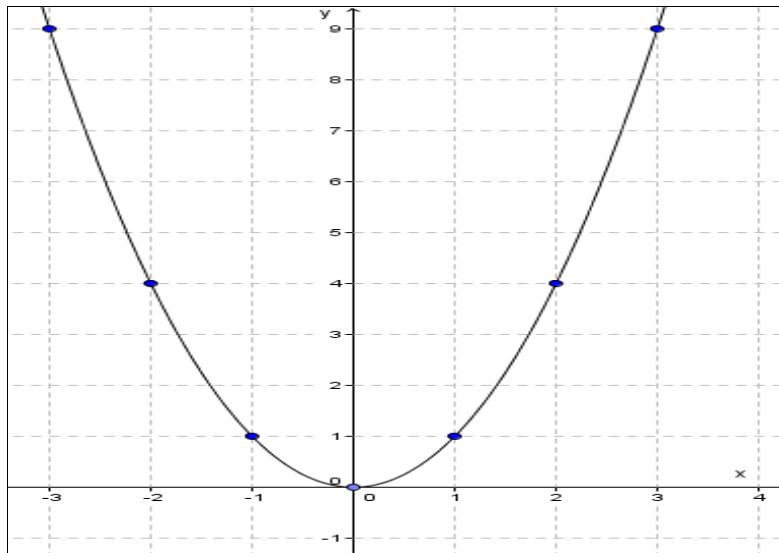
Let $b = c = 0$ and let $a = 1$

Then we have a very simple parabola $y = x^2$

We can use simple substitution to obtain points on this function:

x	0	1	-1	-2	2	-3	3
$y = x^2$	0	1	1	4	4	9	9

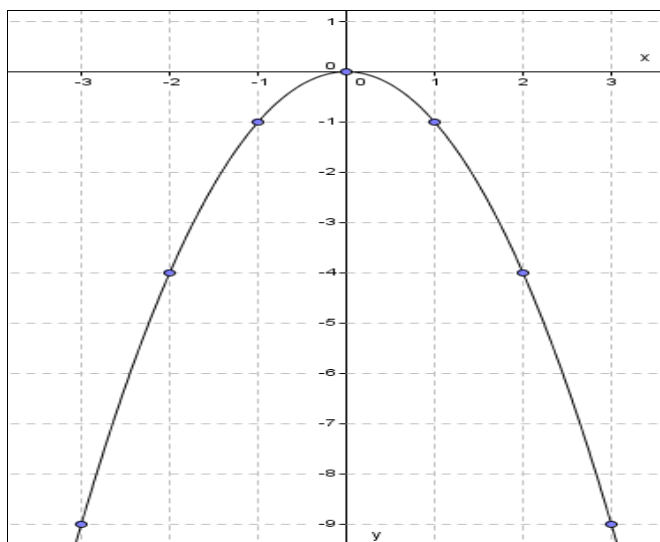
We can plot these points on the Cartesian plane:



NB: The above sketch was drawn using GeoGebra

For the graph of $y = -x^2$ the y values are negative.

x	0	1	-1	-2	2	-3	3
$y = -x^2$	0	-1	-1	-4	-4	-9	-9





Worked Examples 1-3: Simple sketches of quadratic Functions(10 Mins)

The facilitator will now take you through a number of sketches of quadratic functions.

Example 1

1.1 Sketch the graphs of:

$$y = x^2$$

$$y = 2x^2$$

$$y = 3x^2$$

$$y = \frac{1}{2}x^2$$

$$y = \frac{1}{4}x^2$$

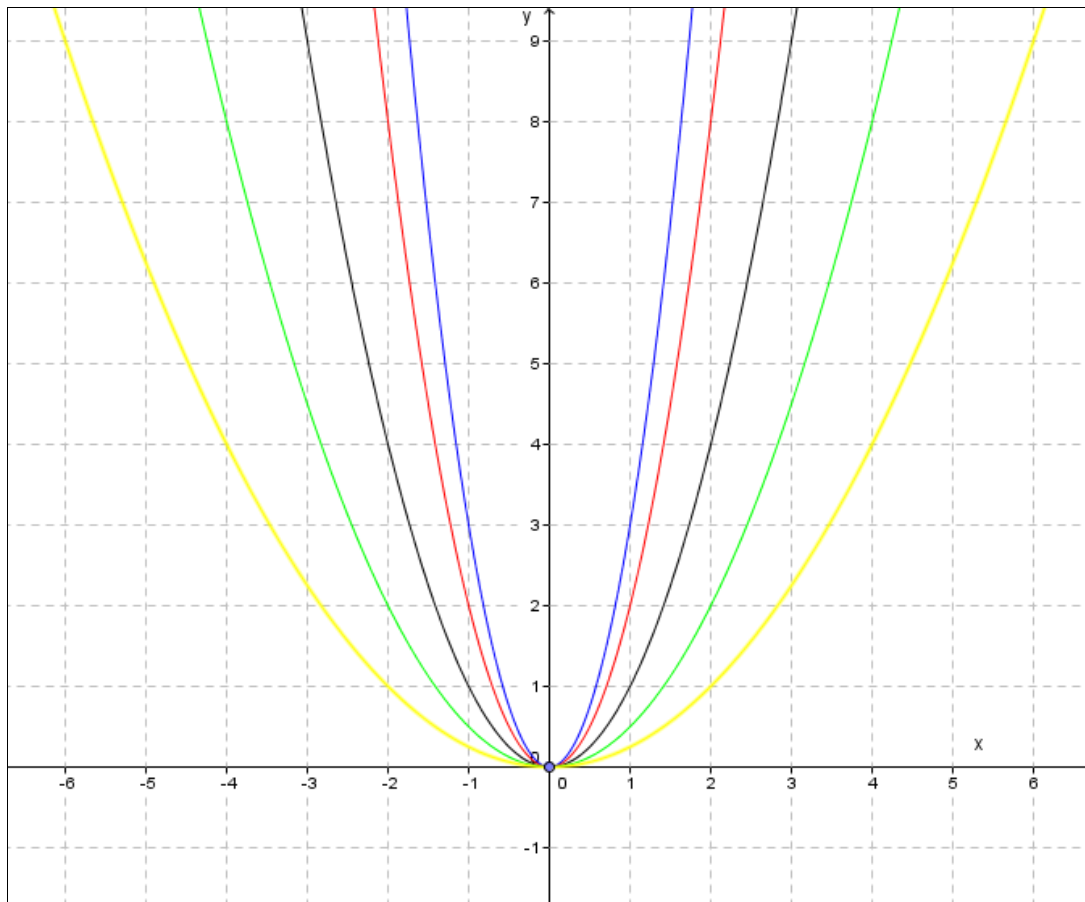
1.2 What do you observe?

Solution

We can bring down the table and extend the number of rows:

x	0	1	-1	-2	2	-3	3
$y = x^2$	0	1	1	4	4	9	9
$y = 2x^2$	0	2	2	8	8	18	18
$y = 3x^2$	0	3	3	12	12	27	27
$y = \frac{1}{2}x^2$	0	$\frac{1}{2}$	$\frac{1}{2}$	2	2	4,5	4,5
$y = \frac{1}{4}x^2$	0	$\frac{1}{4}$	$\frac{1}{4}$	1	1	2,25	2,25

1.1 Graphs:



1.2 If a increases, the graph becomes narrower or stretches. As a decreases the graph becomes wider or flatter.

Example 2

2.1 Sketch the graphs of:

$$y = -x^2$$

$$y = -2x^2$$

$$y = -3x^2$$

$$y = -\frac{1}{2}x^2$$

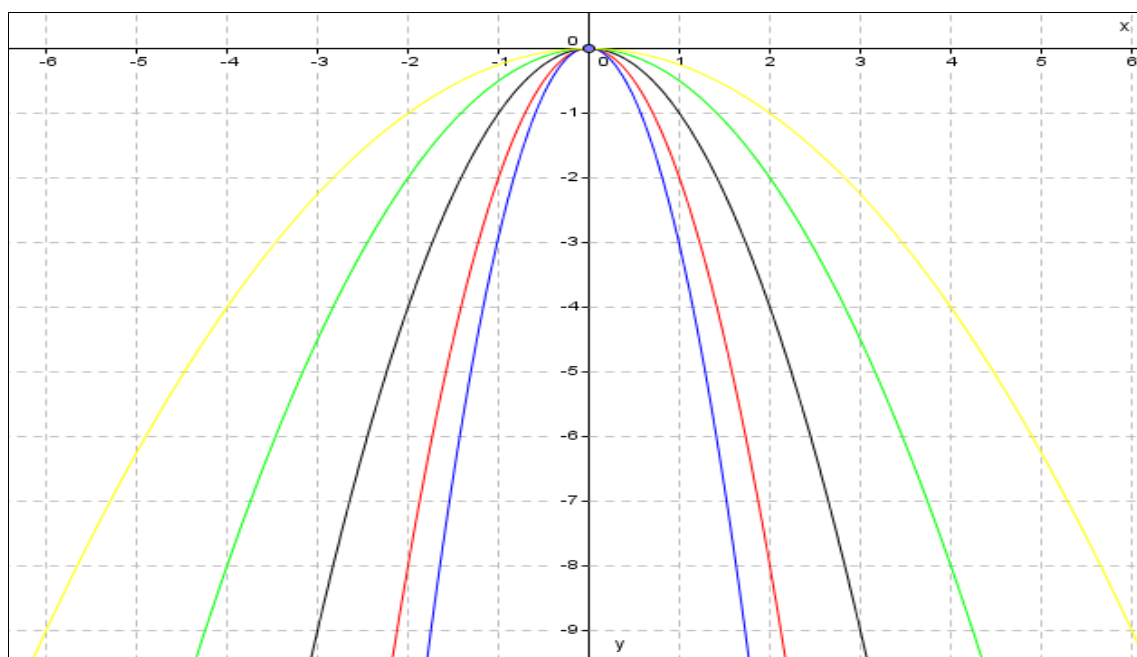
$$y = -\frac{1}{4}x^2$$

2.2 What do you observe?

Solution

2.1 We can bring down the table and extend the number of rows:

x	0	1	-1	-2	2	-3	3
$y = -x^2$	0	-1	-1	-4	-4	-9	-9
$y = -2x^2$	0	-2	-2	-8	-8	-18	-18
$y = -3x^2$	0	-3	-3	-12	-12	-27	-27
$y = -\frac{1}{2}x^2$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-2	-2	-4,5	-4,5
$y = -\frac{1}{4}x^2$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	-1	-1	-2,25	-2,25



2.2 As a decreases the graph becomes narrower or stretches. As a increases the graph becomes wider or flatter.

Some properties of the function $f = \{(x;y) : y = ax^2\}$

- When $a > 0$, it has a minimum value. When $a < 0$, it has a maximum value
- The y -axis ($x = 0$) is the axis of symmetry
- The turning point is $(0;0)$
- Domain = $x: x \in R$
- When $a > 0$, the range = $\{y: y \geq 0; y \in R\}$
- When $a < 0$, the range = $y: y \leq 0; y \in R$

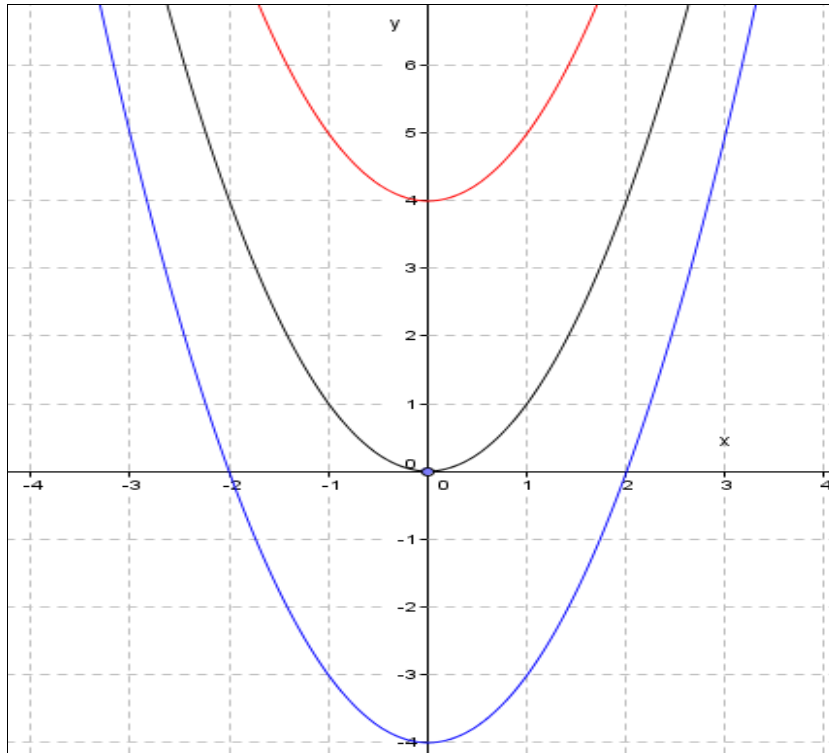
Example 3

3.1 Sketch the graphs of

$$y = x^2$$

$$y = x^2 + 4$$

$$y = x^2 - 4$$



Activity 1-3: Basic exercise

Group organisation:

Time:

Resources:

Appendix:

Pairs

30 min

• Graph Paper

None

1. Read each question carefully.
2. Discuss the question with your partner and proceed to answer the questions. You may draw the graphs on the graph paper supplied and answer the questions alongside the graphs.

1. Given the function:

$$f = \{(x; y) : y = -2x^2\}$$

- 1.1 Sketch the graph of f .

- 1.2 If f is moved 3 units downwards, to form the graph of g , determine the equation defining graph g and its maximum value.

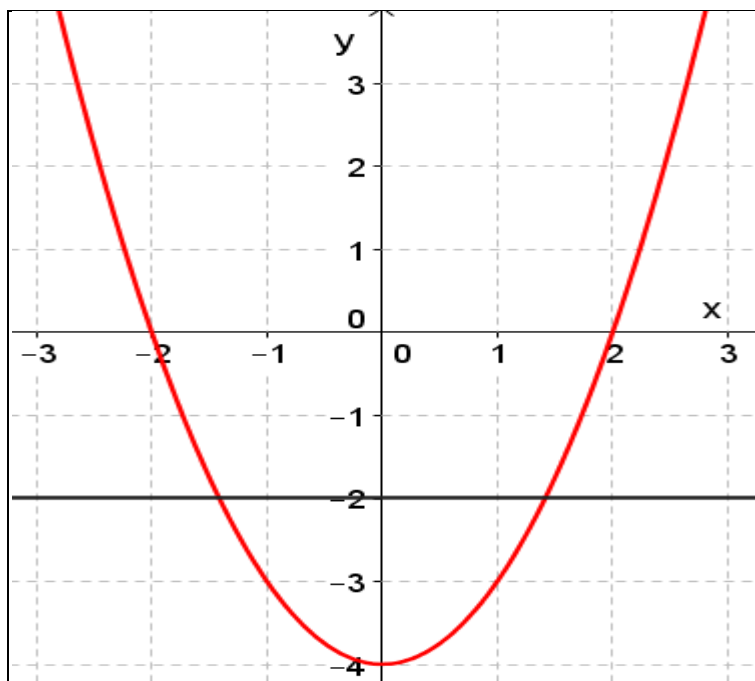
- 1.3 Describe how the graph of $y = x^2$ may be transformed firstly to f and then to g .

2. Sketch the graphs of $g = \{(x; y) : y = 2x^2\}$ and $h = \{(x; y) : y = x + 1\}$ on the same system of axes and determine the points of intersection.
3. Given $f = \{(x; y) : y = -2x^2 + 8\}$
 - 3.1 Determine the y-intercept (that is, the value of $f(0)$).
 - 3.2 Determine the x-intercepts.
 - 3.3 Does f have a maximum or minimum value? Explain your answer.
 - 3.4 Write down the coordinates of the turning point.
 - 3.5 Use the information in 3.1 to 3.4 to draw the graph of $f = \{(x; y) : y = -2x^2 + 8\}$
 - 3.6 Join your y-intercept to your negative x-intercept. Call this line g . Determine the equation of g .

Enrichment

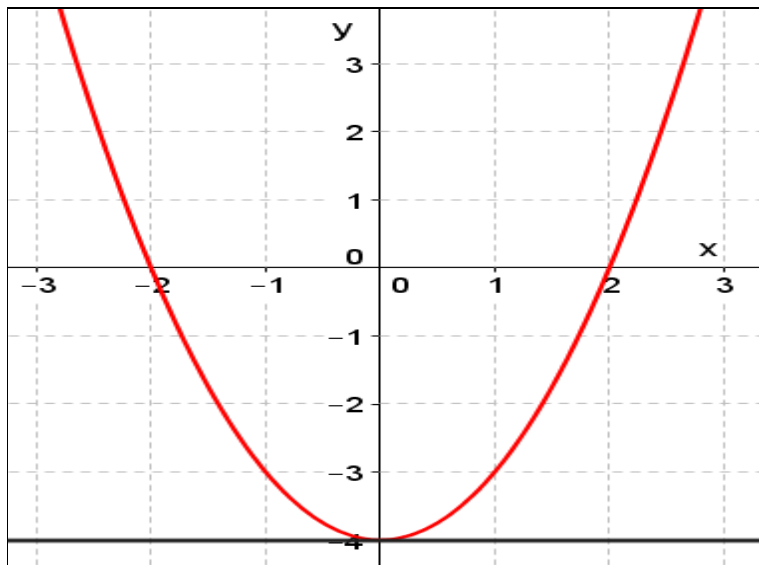
Parabolas and intersecting lines:

The graph of $y = x^2 - 4$ and $y = -2$ are drawn below:



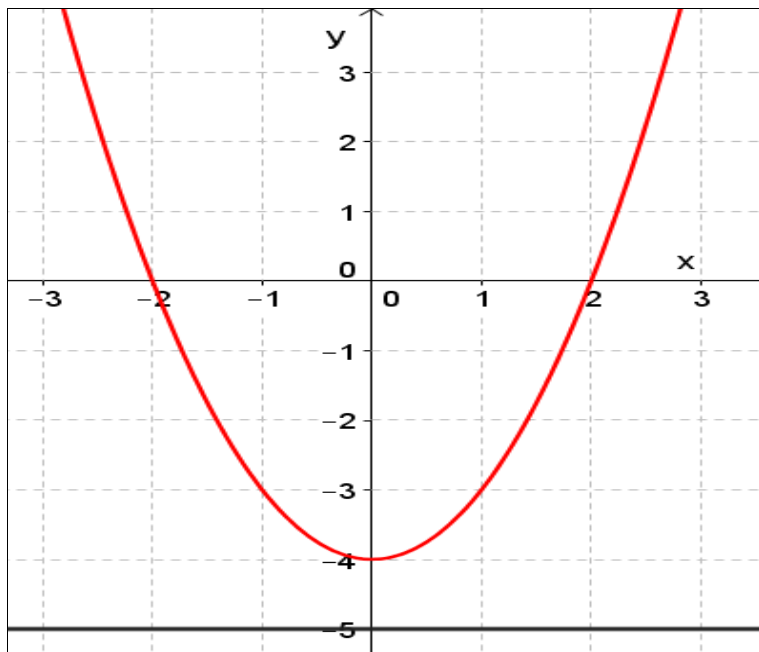
We note that there are 2 points of intersection.

We can move $y = -2$ down 2 units to get $y = -4$.



$y = -4$ is a tangent to $y = x^2 - 4$.

If we move $y = -4$ further down to say $y = -5$ then we have:



There are no points of intersection.

Question:

The graphs of $y = x^2 - 4$ and $y = d$ are drawn. For what value(s) of d will there be:

- (a) Two points of intersection? (two roots)
- (b) One point of intersection? (one root)
- (c) No points of intersection? (no roots)

Solution

We get the value(s) of d from the graphs

- (a) $d > -4$
- (b) $d = -4$
- (c) $d < -4$

This question can be phrased in another way. See below:

Use your graph to determine the value(s) of d for which the roots of $x^2 - 4 = d$

- (a) Real and unequal (two points of intersection).
- (b) Real and equal (one point of intersection).
- (c) Non-real or imaginary (no points of intersection).

Misconceptions

Learners must know the following:

- If the y-intercept of a graph is 6, then the coordinate of the point is (0;6).
- If the x-intercepts are -2 and 4, then the coordinates are (-2;0) and (4;0).

Conclusion

The work done in this module only included examples of parabolas which had the y-axis as the axis of symmetry. The graphs drawn were translated up and down the y-axis. These examples provide more than suitable preparation for the ones that follow in the next module.

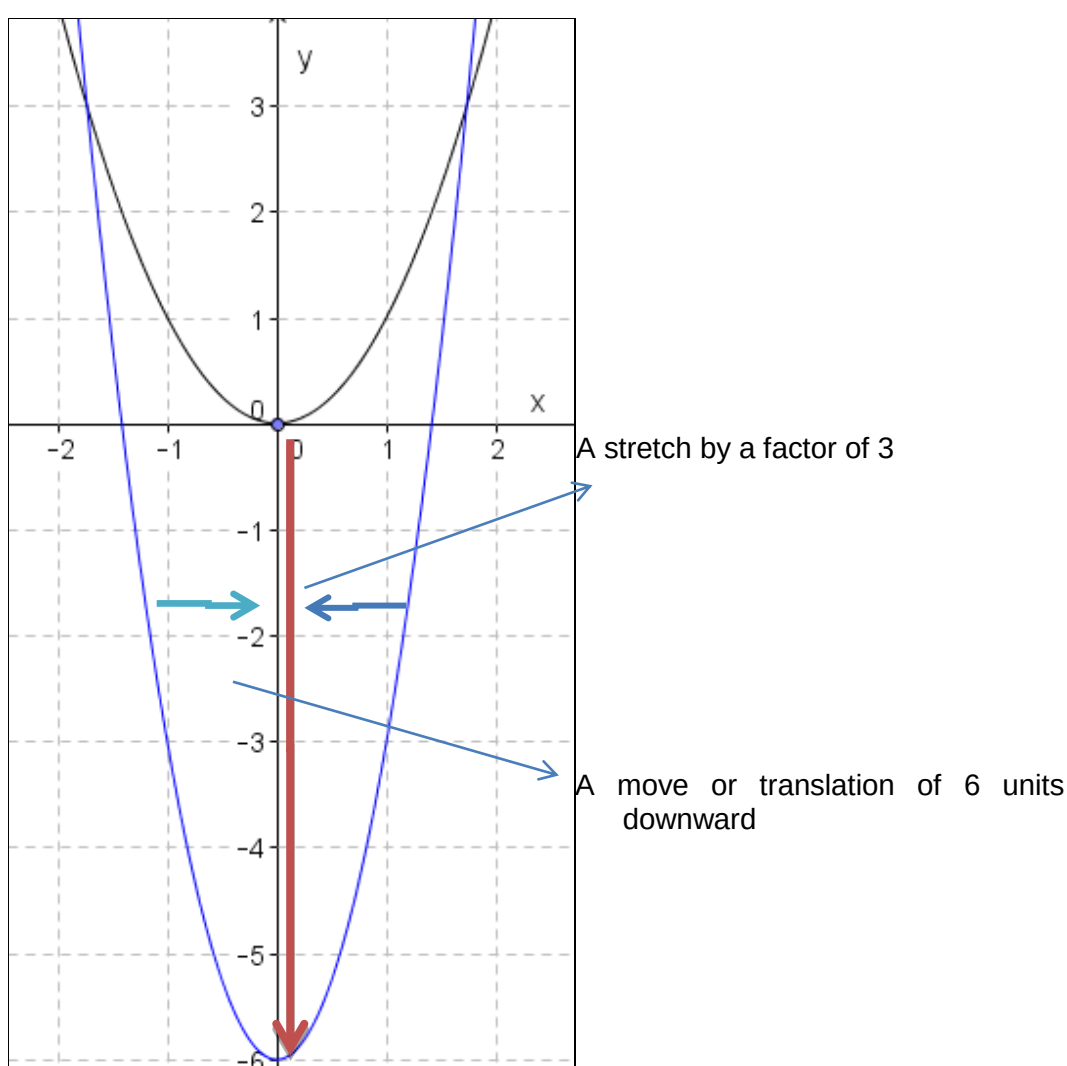
Sub-topic 3: Sketching quadratic functions

Introduction

We have seen from the previous module that the graph of $\{(x;y) : y = ax^2 + q\}$ can be stretched or flattened depending on the value of a . The value of q will dictate whether the graph moves up or down the y -axis.

For example the graph of $g = \{(x;y) : y = 3x^2 - 6\}$ tells us that $y = x^2$ has been stretched by a factor of 3 and moved 6 units down on the y -axis.

We can see this in the graph below:



We could also move $y = ax^2$ along the x -axis (left or right)

The graph of $y = a(x - p)^2$



Worked Examples 1-3: Simple sketches of quadratic Functions(30 Mins)

Pay careful attention as the facilitator discusses the various sketches of quadratic functions.

Don't forget to ask questions and make notes.

Example 1

1.1 Sketch the following graphs on the same system of axes:

$$y = x^2$$

$$y = (x - 2)^2$$

$$y = (x + 3)^2$$

1.2 What do you notice?

Solution

1.1 Graphs



- 1.2 We also note that $y = (x + 3)^2$ is the graph of $y = x^2$ moved 2 units to the right (along the x-axis). The turning point of $y = (x + 3)^2$ is $(-3;0)$. We note that $y = (x - 2)^2$ is the graph of $y = x^2$ moved 3 units to the left (along the x-axis). The turning point of $y = (x - 2)^2$ is $(2;0)$. Please make note of how the turning points change.

Example 2

- 2.1 Sketch the following graphs on the same system of axes:

$$y = x^2$$

$$y = -x^2$$

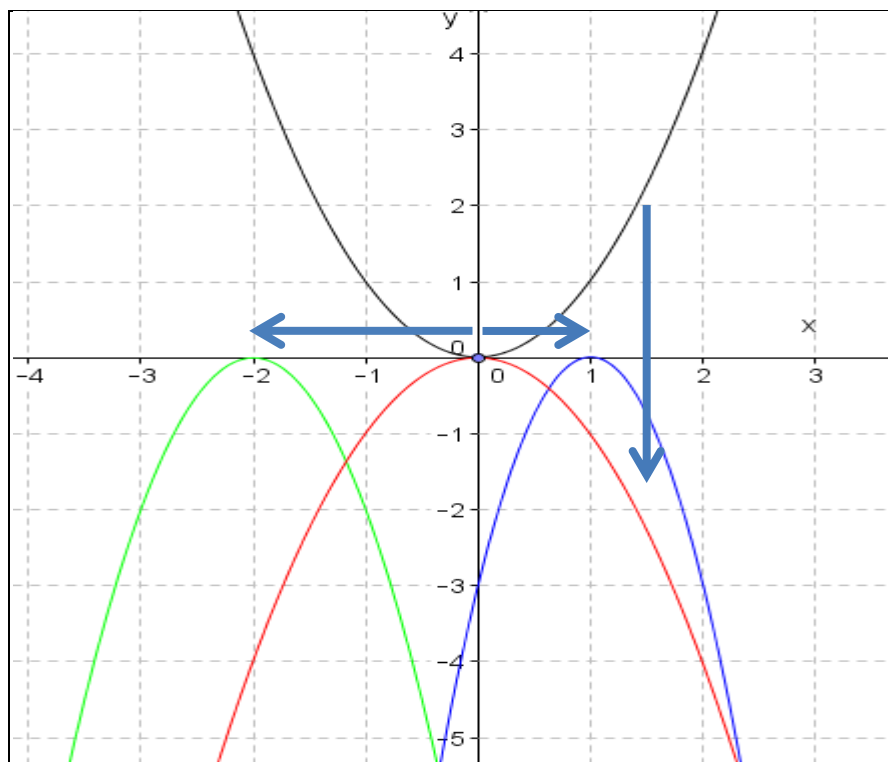
$$y = -3(x - 1)^2$$

$$y = -2(x + 2)^2$$

- 2.2 What do you notice?

Solution

- 2.1 Graphs



2.1 The graph of $y = -x^2$ is a reflection of $y = x^2$ in the x-axis. The graph of $y = -3(x-1)^2$ is a stretch of $y = -x^2$ by a factor of 3, moved 1 unit to the right. The graph of $y = -2(x+2)^2$ is a stretch of $y = -x^2$ by a factor of 3, moved 2 units to the left.

Working with $\{(x;y) : y = a(x+p)^2 + q\}$

Consider the example: $f = \{(x;y) : y = 2(x-1)^2 - 8\}$

Without drawing the graph of $f = \{(x;y) : y = 2(x-1)^2 - 8\}$ we can make some deductions about its properties:

Firstly $y = 2(x-1)^2 - 8$ is a move of $y = x^2$ 1 unit to the right, then a stretch by a factor of 2, followed by a move of 8 units downward.

The turning point of $y = 2(x-1)^2 - 8$ is (1;-8). **Do you know why this is the case?**

Now

$$\begin{aligned} y &= 2(x-1)^2 - 8 \\ &= 2(x^2 - 2x + 1) - 8 \\ &= 2x^2 - 4x + 2 - 8 \\ &= 2x^2 - 4x - 6 \end{aligned}$$

Thus, $y = 2x^2 - 4x - 8$ is another way of writing $y = 2(x-1)^2 - 8$.

How do we transform $y = 2x^2 - 4x - 6$ to $y = 2(x-1)^2 - 8$?

We do it as follows:

$$y = 2x^2 - 4x - 6$$

We can use the method of "completing the square"

$$\begin{aligned} y &= 2(x^2 - 2x) - 6 \quad (\text{take coefficient of } x^2 \text{ as common factor for first 2 terms}) \\ &= 2(x^2 - 2x + 1 - 1) - 6 \quad (\text{add and subtract } \left(\frac{1}{2} \text{ coefficient of } x\right)^2 \text{ within brackets}) \\ &= 2(x-1)^2 - 8 \quad (\text{first three terms form a perfect square; multiply 4th term by common factor and take outside}) \end{aligned}$$

Example 3

3. Given the following:

$$f = \{(x; y) : y = 2x^2 - 4x - 6\} = \{(x; y) : y = 2(x - 1)^2 - 8\}$$

- 3.1 Determine the x and y-intercepts.
- 3.2 Write down the coordinates of the turning point.
- 3.3 Write down the equation of the axis of symmetry.
- 3.4 Does the graph have a minimum or maximum value? Explain your answer.
- 3.5 Draw the graph of f.

Solution

3.1 y intercept is $f(0) = -6$

x-intercepts:

$$2x^2 - 4x - 6 = 0$$

$$\Leftrightarrow x^2 - 2x - 3 = 0 \quad (\text{Divide by 2})$$

$$\Leftrightarrow (x + 1)(x - 3) = 0$$

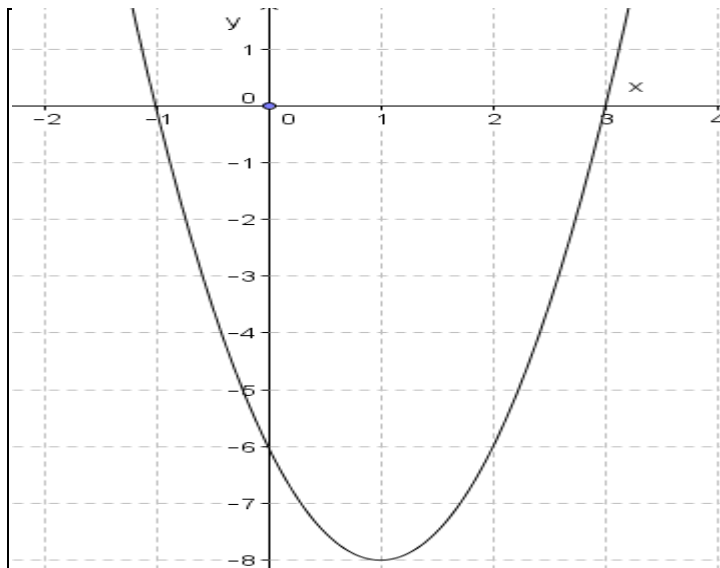
$$\Leftrightarrow x = -1 \text{ or } 3$$

3.2 Turning point (1; -8)

3.3 $x = 1$

3.4 The graph has a minimum value since $a = 2$ (positive).

3.5 Graph



In general, the following will apply:

If $f = \{(x; y) : y = ax^2 + bx + c\}$ then:

(1) Axis of symmetry: $x = -\frac{b}{2a}$

(2) Minimum or maximum value: $y = \frac{4ac - b^2}{4a}$

(3) Turning point $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

(4) y- intercept occurs when $x = 0$. Thus; $y = c$ (is the y-intercept).

(5) x-intercepts occur when $y = 0$, so $ax^2 + bx + c = 0$. This is solved by factorisation or using the formula.

NB: Gradients and average gradients

We know learners should be familiar with straight line graphs from grade 9.

The general form of a straight line function is $\{(x,y) : ax + by = c\}$

$$\text{If } ax + by + c = 0$$

$$\text{then } by = -ax - c$$

$$\Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$$

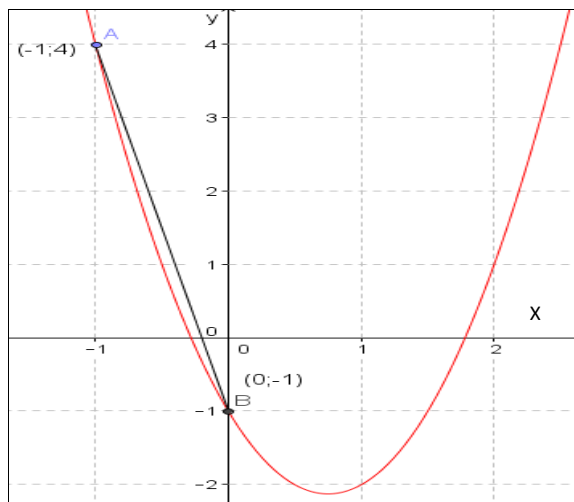
$$\Rightarrow y = mx + k \quad \left[-\frac{a}{b} = m; -\frac{c}{b} = k \right]$$

where m is the gradient and k is the y – intercept.

Thus, in an equation such as $y = 2x + 3$ the gradient ($m = 2$) is the same throughout.

However, in a parabola, the gradient is changing all time. We can, however, find the average gradient between two points on the parabola.

Given: $f(x) = 2x^2 - 3x - 1$



The points A (-1;4) and B(0;-1) lie on the graph. The average gradient of the line joining A to B is calculated as follows:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(0) - f(-1)}{0 - (-1)} = \frac{-1 - 4}{1} = -5$$



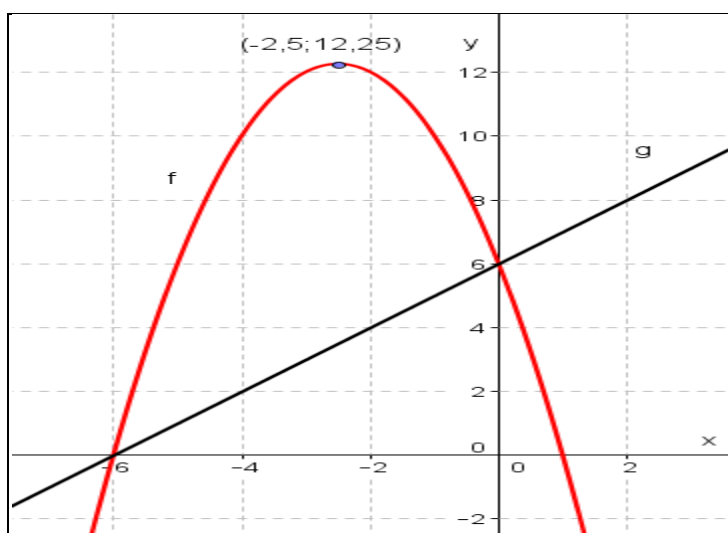
Activity 1: Graphs and Inequalities

Group organisation:	Time:	Resources:	Appendix:
Individual	5 min	• Participant Handout	None

1. Read each question carefully.
2. Discuss the question with your partner and proceed to answer the questions in the space provided below.

The graphs of $f(x) = -x^2 - 5x + 6$ and $g(x) = x + 6$ are drawn below.

The intercepts with the axes $(-6;0)$; $(1;0)$; $(0;6)$ and the turning point $(-2,5;12,25)$ are shown.



Determine the following:

The value(s) of x for which:

(a) $f(x) \geq g(x)$

(b) $f(x) \cdot g(x) \leq 0$

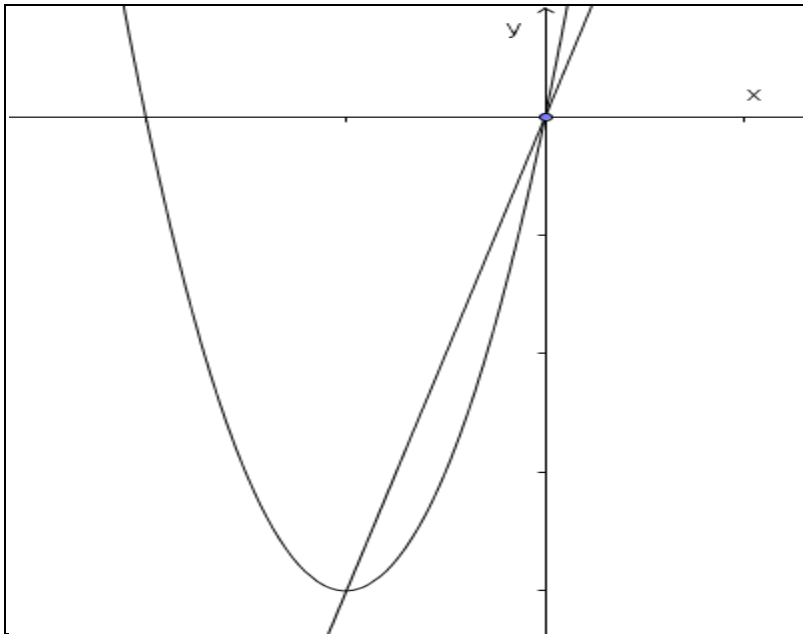


Activities 2-4: Graphs

Group organisation:	Time:	Resources:	Appendix:
Individual	45 min	• Flipchart paper • Whiteboard markers	None
In your groups you will: 1. Select a scribe and a spokesperson for this activity only.(Rotate from activity to activity). 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

2. Given: $f = \{(x; y) : y = -2x^2\}$
 - 2.1 Sketch the graph of f .
 - 2.2 The graph of f is moved 3 units upward and 2 units to the right to form the graph g . Draw the graph of g on the same system of axes as f .
 - 2.3 Write down the defining equation of g .
3. Sketch the graphs of $f = \{(x; y) : y = -2x^2 + 4x\}$ and $g = \{(x; y) : y = -2x + 2\}$ on the same system of axes. The determine:
 - 3.1 The value/s of x for which $f(x) < 0$
 - 3.2 The value/s of x for which $g(x) \geq f(x)$

4. Given the graphs of $f = \{(x; y) : y = 2x(x + 4)\}$ and $g = \{(x; y) : y = 4x\}$ drawn below:



- 4.1 Determine the value/s of x for which $f(x) = 0$.
- 4.2 Calculate P the turning point of f .
- 4.3 Determine the average gradient of the curve of f between $x = -3$ and 0 .
- 4.4 Hence, state what you can deduce about the function f between -3 and 0 .
- 4.5 Determine the value/s of x for which $f(x) > 0$.
- 4.6 Give the coordinates of the turning point of $f(x+3)-1$.
- 4.7 For $h(x) = 4x - 2x(x - 4)$, determine the maximum value of h . For what value of x does this maximum occur?

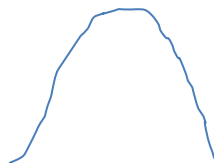
Enrichment

Given $f(x) = ax^2 + bx + c$ with $a < 0$; $b > 0$ and $c > 0$.

Draw a rough sketch of $f(x)$

$$a < 0$$

The graph has a maximum value:



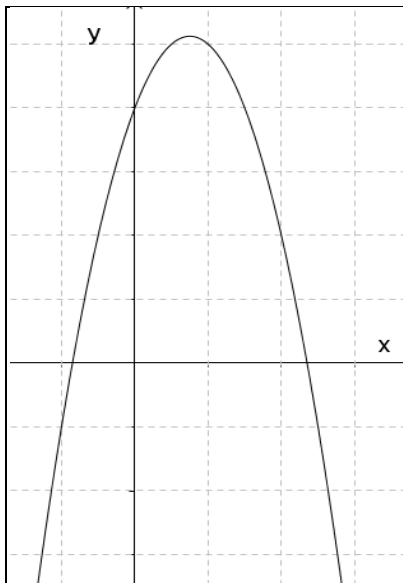
$$a < 0; b > 0 \Rightarrow -\frac{b}{2a} \text{ is positive (why)} \left(-\frac{+}{-} \right)$$

Thus the axis of symmetry is positive

$$c > 0$$

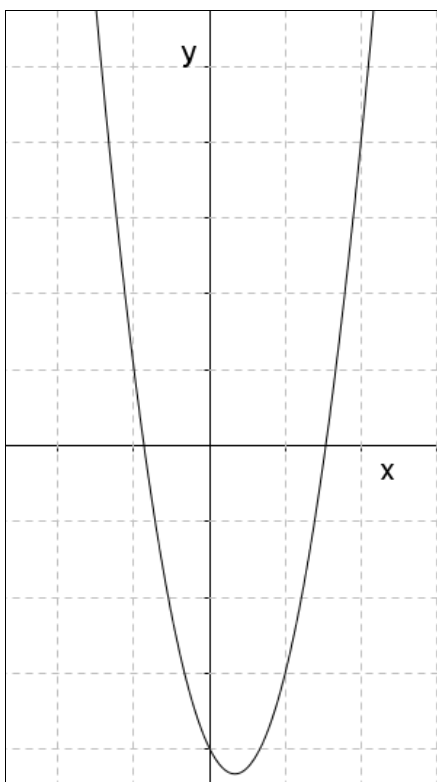
The y -intercept is positive.

So a rough sketch will look as follows:



Given $g(x) = ax^2 + bx + c$ with $a > 0$; $b < 0$ and $c < 0$ and x-intercepts of different signs. Draw a sketch graph of g
 Draw a rough sketch of $f(x)$

Possible graph

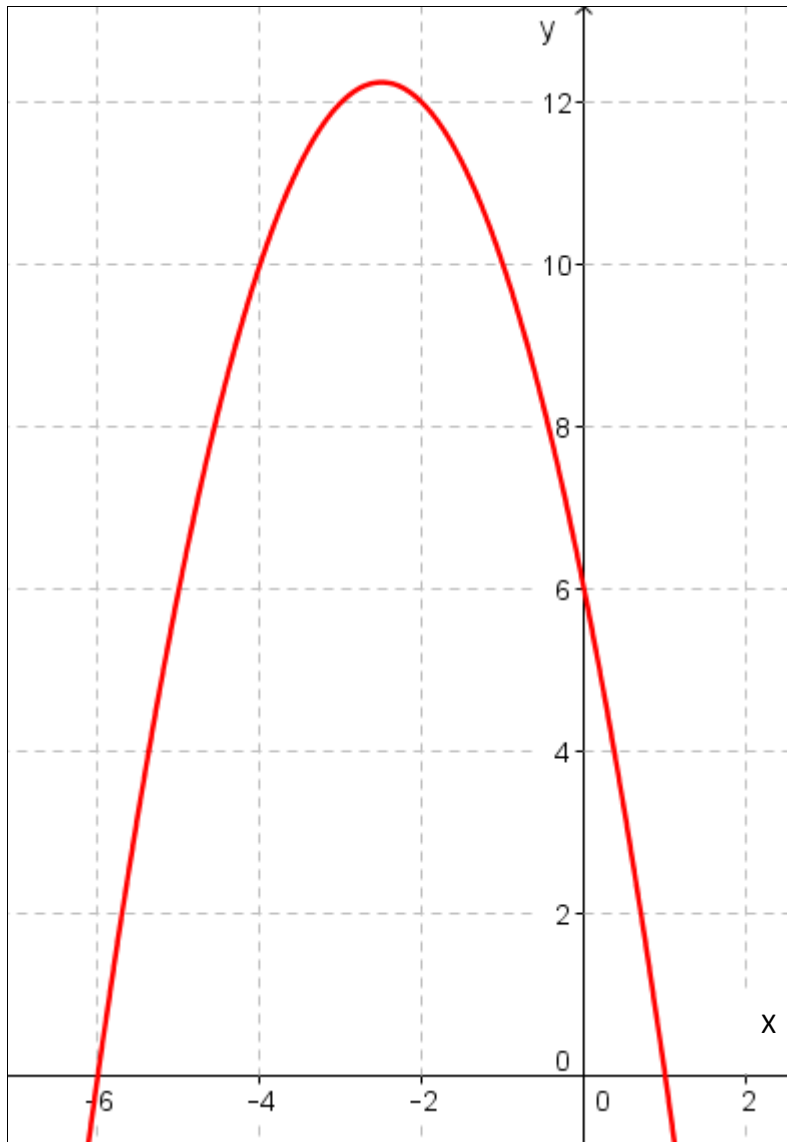


In module 2 we saw how quadratic functions and a straight line graph can be used to determine roots of quadratic equations. Let us do another example:

Suppose the following question was given:

Use your graph(s) to find real value(s) of k for which $-x^2 - 5x + 6 = k$ has two negative, unequal roots.

We draw the graph of $y = -x^2 - 5x + 6$



The maximum value is found in two ways:

We find the axis of symmetry and then substitute in the function to find the y value.

$$x = -\frac{b}{2a} = -\frac{(-5)}{2(-1)} = -2.5$$

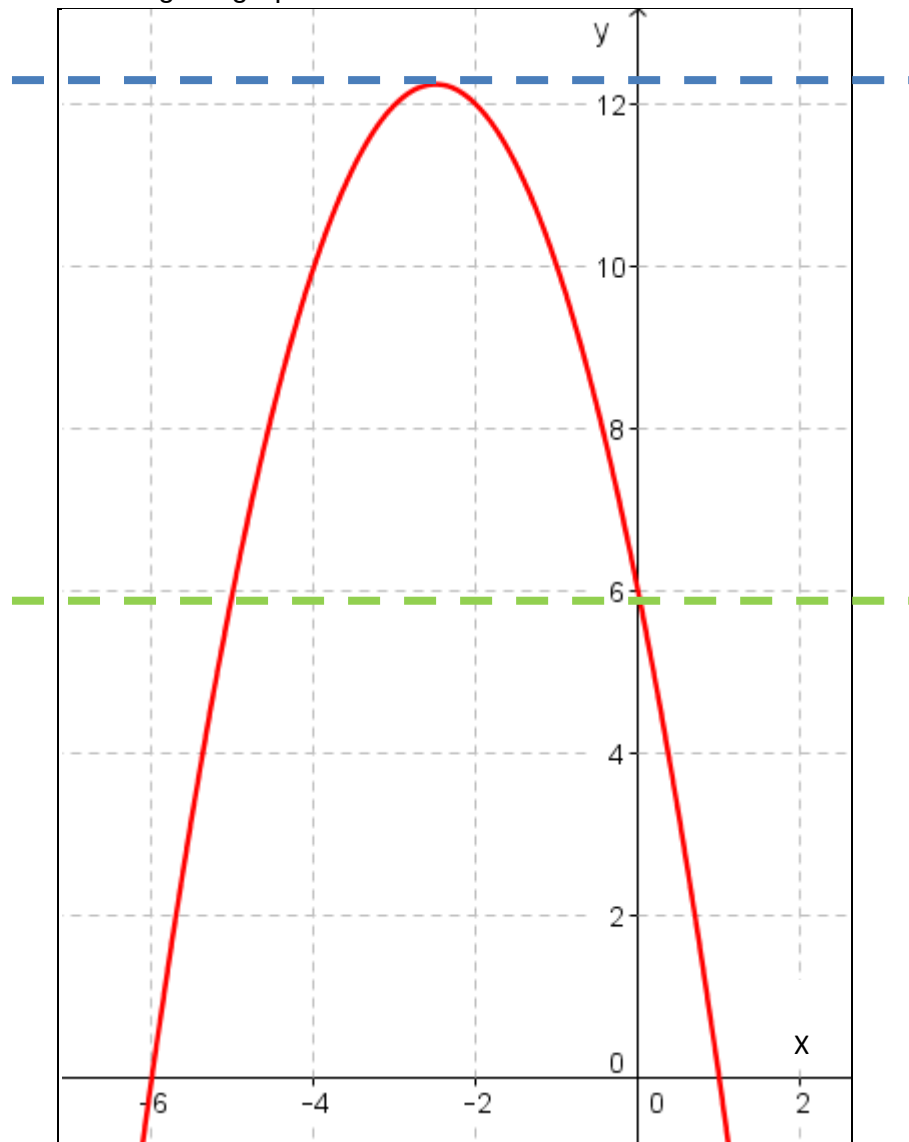
$$y = -(-2.5)^2 - 5(-2.5) + 6 = 12.25$$

The maximum value is $y = 12,25$.

Another way of find the maximum value is:

$$y = \frac{4ac - b^2}{4a} = \frac{4(-1)(6) - (-5)^2}{4(-1)} = \frac{-49}{-4} = 12,25$$

We bring the graph down:



The value of k must lie between the y - intercept ($y = 6$) and the maximum value $y = 12,25$ for there to be two negative, unequal roots.

Thus, $6 \leq y \leq 12,25$

Misconceptions

Highlight the following:

- The importance of correctly plotting points on the Cartesian Plane.
- Knowing the difference between the parabola and the cubic function

How to work out:

$f(0) \rightarrow$ the y-intercept and its difference with $f(x) = 0$ (calculation of the x-intercepts)

Conclusion

When working with your learners please point out the following:

- The need to practise working with graphs with both maximum and minimum values.
- Being able to interpret the different features of graphs.
- The intersection between a quadratic function (parabola) and a straight line and its link to nature of roots.
- How to work out the product of functions and inequalities.

Sub-topic 4: The hyperbola

Grade 10	Grade 11	Grade 12
Point-by point plotting of $y = \frac{1}{x}$ Shape of functions, domain & range; axis of symmetry; turning points, intercepts on the axes The effect of a and q on functions defined by $y = a.f(x) + q$ where $f(x) = \frac{1}{x}$ Sketch graphs and find the equation of graphs and interpret graphs	Revise the effect of the parameters a and q and investigate the effect of p and the function defined by $y = \frac{a}{x + p} + q$ Sketch graphs and find the equation of graphs and interpret graphs	

Introduction



Accessed from: <http://www.pleacher.com/mp/mlessons/calculus/apphyper.html>

The above diagram is Dulles Airport in the USA. It was designed by Eero Saarinen and is in the shape of a hyperbolic paraboloid. The hyperbolic paraboloid is a three-dimensional curve that is a hyperbola in one cross-section, and a parabola in another cross section.

Now that we know what shape it is, let us find out more about the shape and properties of the hyperbola. We start with the general form of the hyperbola:

The general form of the hyperbola is given by:

$$h = \left\{ (x; y) : y = \frac{a}{x + p} + q \right\} \text{ where } a \in R \text{ and } p, q \in R$$

Let $p = 0$ and $q = 0$

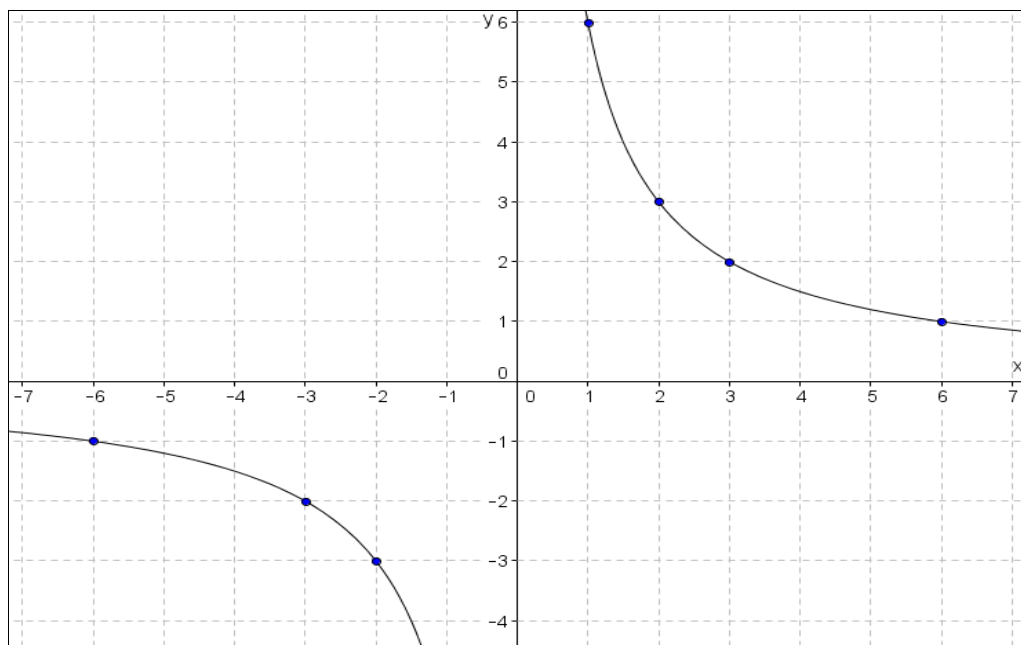
So we have: $y = \frac{a}{x}$

If $a = 6$ so $y = \frac{6}{x} \Rightarrow xy = 6$

We can use a table to get both x and y values which satisfy this function:

x	1	2	3	6	-1	-2	-3	-6
y	6	3	2	1	-6	-3	-2	-1

We now use point-by-point plotting to draw the graph of $xy = 6$ or $y = \frac{6}{x}$. Please note that both negative and positive values are used. We also join the points.





Worked Examples 1- 4: Drawing a hyperbola(30 Mins)

Pay careful attention as the facilitator discusses the sketches of the various hyperbolas.

Don't forget to ask questions and make notes.

Example 1

When $p = 0$ in the function h we have:

$$y = \frac{a}{x} + q$$

1.1 Sketch on the same system of axes, the graphs of:

$$y = \frac{6}{x}$$

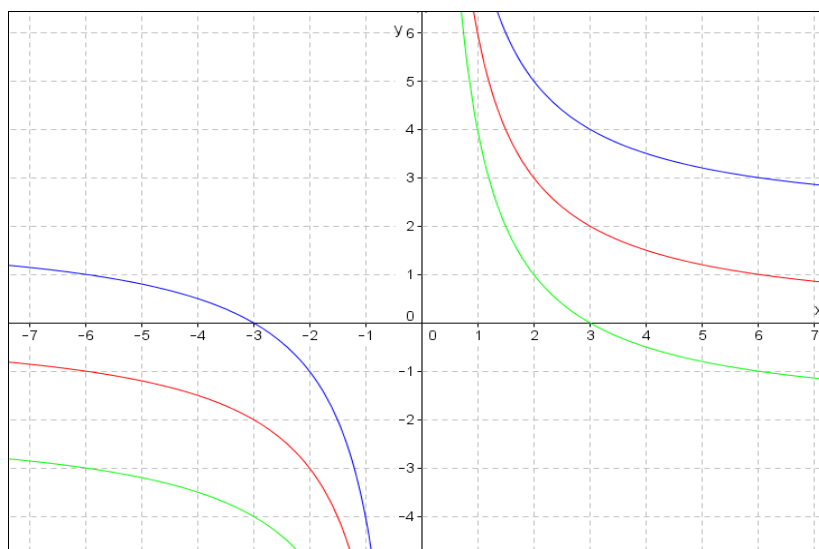
$$y = \frac{6}{x} + 2$$

$$y = \frac{6}{x} - 2$$

1.2 What do you observe?

Solution

1.1 We use Geogebra to sketch the graphs:



1.2 We note that $y = \frac{6}{x} + 2$ is a translation of $y = \frac{6}{x}$ two units upwards (along the y-axis).

We also note that $y = \frac{6}{x} - 2$ is a translation of $y = \frac{6}{x}$ two units downwards (along the y-axis).

Example 2

Let $q = 0$ in the function h . We have $y = \frac{a}{x+p}$

2.1 Sketch (on the same system of axes):

$$y = \frac{6}{x}$$

$$y = \frac{6}{x-1}$$

$$y = \frac{6}{x-2}$$

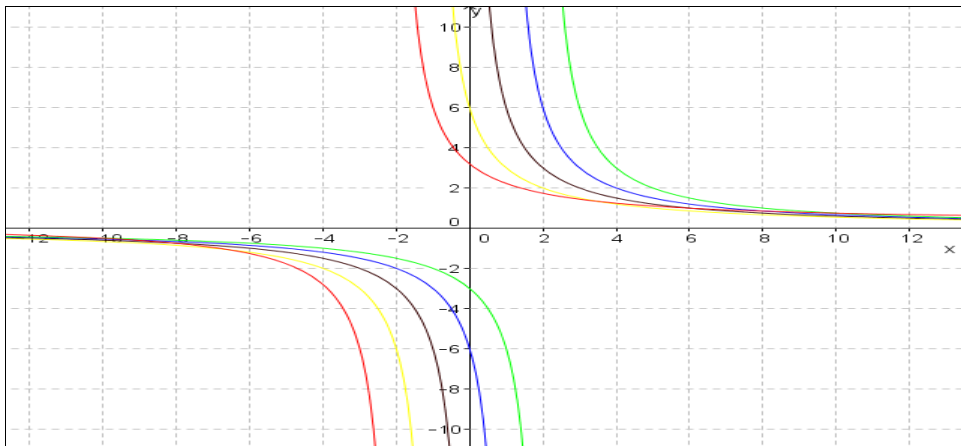
$$y = \frac{6}{x+1}$$

$$y = \frac{6}{x+2}$$

2.2 What do you observe?

Solution

2.1 Once again we use Geogebra to draw these graphs



2.2 We note the following:

- The graph of $y = \frac{6}{x}$ does not cut the x-or y-axis.
- The others cut the y-axis but not the x-axis.
- The x-axis is a **horizontal asymptote** for all the graphs; the others have **vertical asymptotes at different places** ($x = -2$; $x = -1$; $x = 0$; $x = 1$; $x = 2$).
- The graph of $y = \frac{6}{x+1}$ is the graph of $y = \frac{6}{x}$ shifted 1 unit to the left.
- The graph of $y = \frac{6}{x-2}$ is the graph of $y = \frac{6}{x}$ shifted 2 units to the right.
- In general if $y = \frac{a}{x+p}$, then the horizontal asymptote is the x-axis ($y=0$); while the vertical asymptote is when $x = -p$

Example 3

3. Consider the most general form of the hyperbola: $y = \frac{a}{x+p} + q$

Let $y = \frac{6}{x+1} - 2$

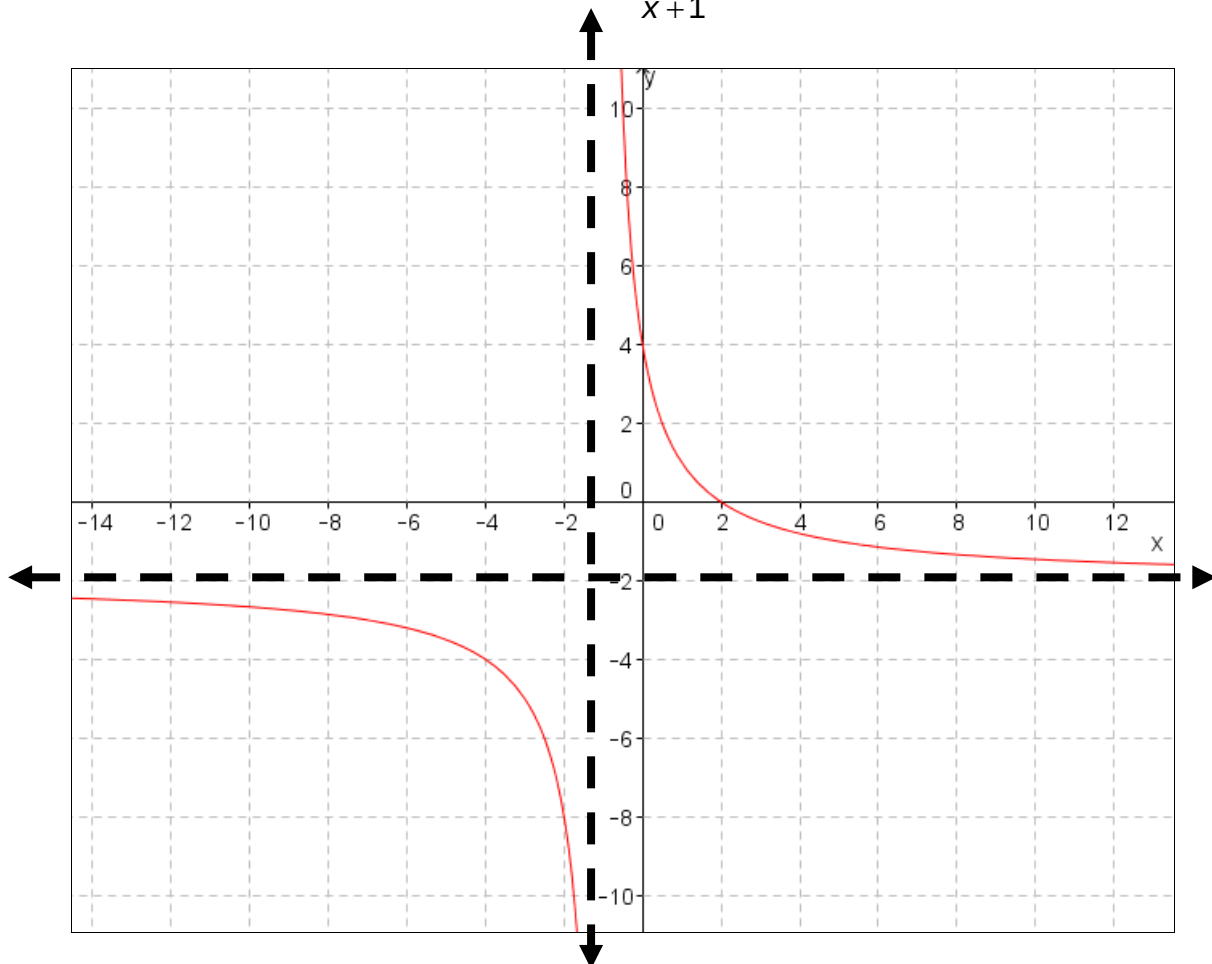
The vertical asymptote is calculated as follows:

$$x+1=0$$

$$\Rightarrow x = -1$$

The horizontal asymptote is the shift (the value of q). In this case it is $y = -2$.

We can see this in the drawn graph of $y = \frac{6}{x+1} - 2$ (below)



We now see the graph of $y = \frac{6}{x+1} - 2$

The thick-broken lines show the asymptotes.

Note: If $a > 0$ then our curves are usually in the first and third quadrants.

If $a < 0$ then we work in the 2nd and 4th quadrants.

Example 4

4.1 Sketch the graphs of:

$$y = \frac{-4}{x}$$

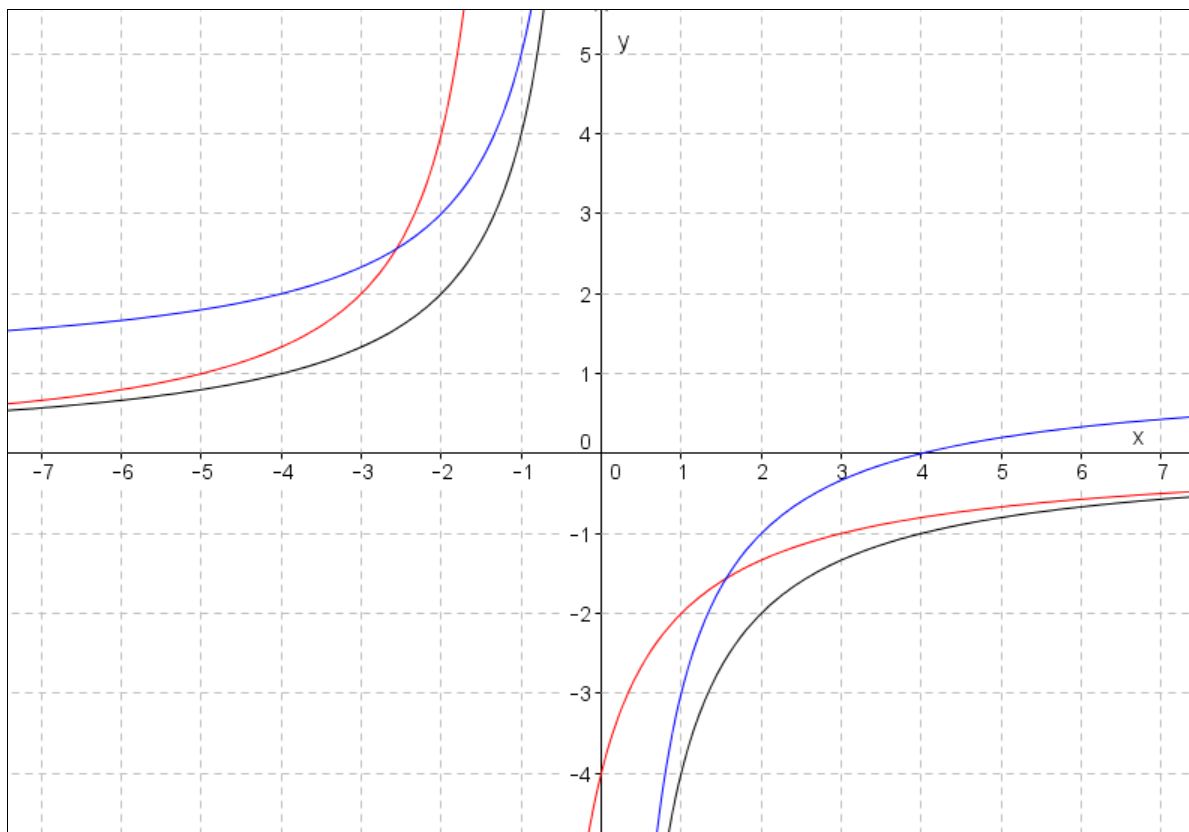
$$y = \frac{-4}{x+1}$$

$$y = \frac{-4}{x} + 1$$

4.2 Determine the asymptotes.

Solution

4.1 Graphs



4.2

$y = \frac{-4}{x} \rightarrow$ vertical asymptote is y -axis; horizontal asymptote is x -axis

$y = \frac{-4}{x+1} \rightarrow$ vertical asymptote is $x = -1$; horizontal asymptote is x -axis

$y = \frac{-4}{x} + 1 \rightarrow$ vertical asymptote is y -axis; horizontal asymptote is $y = 1$

Try to identify these asymptotes on the graph.



Activities 1-4: The Hyperbola

Group organisation:	Time:	Resources:	Appendix:
Pairs(Not the same as before)	60 min	Graph paper and A4 Sheets	None

Together you will:

1. Study the questions and answer them on the graph and A4 paper that is provided.

Activity 1

1. Determine the asymptotes for each of the following functions:

1.1 $y = \frac{4}{x}$

1.2 $y = \frac{4}{x+2}$

1.3 $y = \frac{4}{x} - 2$

1.4 $y = \frac{4}{x+2} - 2$

Activity 2

2. Given $f = \left\{ (x; y) : y = \frac{-8}{x+2} - 2 \right\}$

- 2.1 Determine the domain of f .
- 2.2 Determine the coordinates of the x-and y-intercepts of f .
- 2.3 Write down the equations of the asymptotes.
- 2.4 Draw the graph of f .
- 2.5 Now use your graph to determine the range of f .

Activity 3

3. The graph of $g = \left\{ (x; y) : y = \frac{-10}{x} \right\}$ is translated 3 units to the right and 4 units upwards to form the graph of h . Determine the equation of h

Activity 4

4. Draw the graph of $y = \frac{x+2}{x-1}$

Misconceptions

Learners should be able to calculate the value of p and q from a drawn graph of

$$y = \frac{a}{x+p} + q$$

Learners should know that another form of the hyperbola is

$$y = \frac{x+b}{x+c}$$

eg) $y = \frac{x+2}{x-3}$ can be written as $y = \frac{5+x-3}{x-3} = \frac{5}{x-3} + 1$

Conclusion

Summarise key points of this module:

- The value of a in $y = \frac{a}{x+p} + q$ determines the location of the hyperbola.

If $a > 0$, it will be in the “first” and “third” quadrants; if $a < 0$ it will be in the “second” and “fourth” quadrants.

- The vertical asymptote is calculated as $x + p = 0 \Rightarrow x = -p$
- The horizontal asymptote is $y = q$

Sub-topic 5: The exponential function

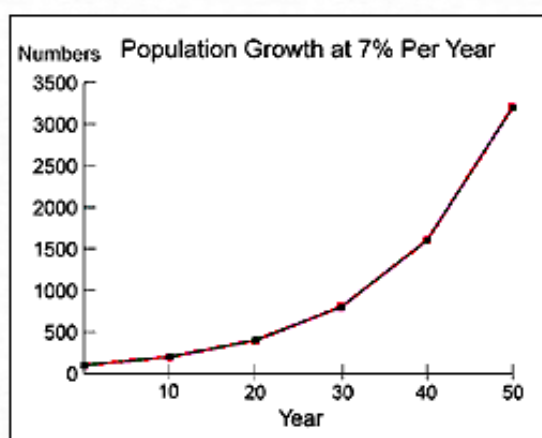
CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
Point-by point plotting of $y = b^x$ Shape of functions, domain & range; axis of symmetry; turning points, intercepts on the axes The effect of a and q on functions defined by $y = a.f(x) + q$ where $f(x) = b^x$ Sketch graphs and find the equation of graphs and interpret graphs	Revise the effect of the parameters a and q and investigate the effect of p and the function defined by $y = a.b^{x+p} + q \quad b > 0; b \neq 1$ Sketch graphs and find the equation of graphs and interpret graphs	

Introduction

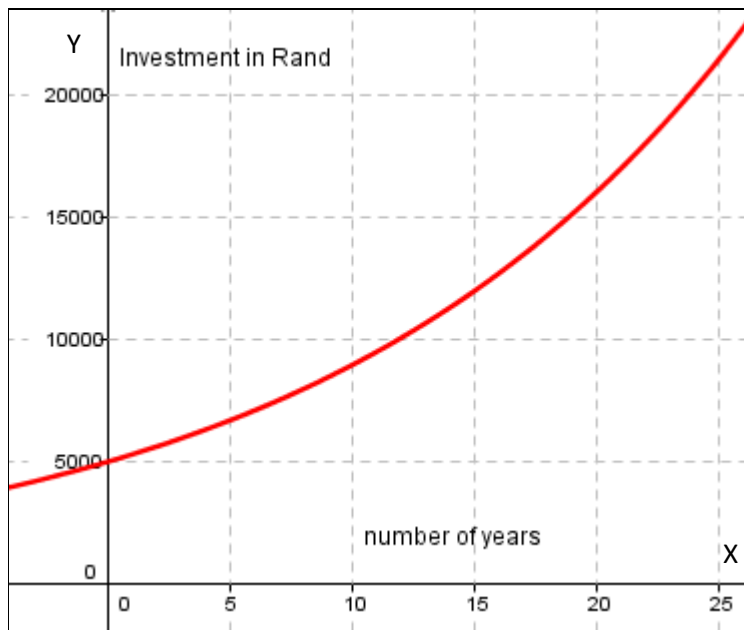
Exponential growth and decay is the rate of change in a graph, whether it is increasing or decreasing.

Real life examples of exponential growth are population, plants, animals and, bacteria.



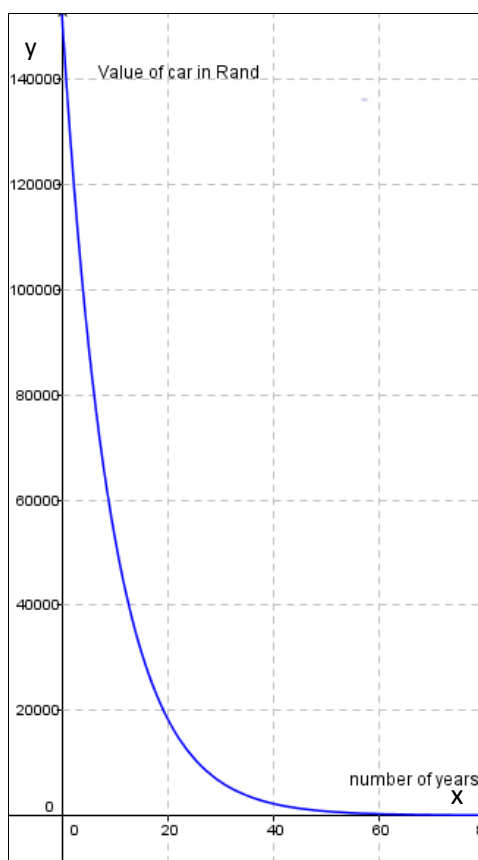
<http://www.ecofuture.org/pop/facts/exponential70.html>

The graph below shows the compound growth on an investment of R5000,00 for n years at 6% per annum, compounded annually.



Examples of exponential decay are population (during genocide), the economy, the ozone layer and the temperature of water.

The graph below shows how a car worth R150,000 depreciates at 10% per annum according to the reducing balance method. This is an example of compound decay.



Thus, one can see why it is important to study the exponential function.

The exponential function $\{(x; y) : y = ab^{x+p} + q; b > 0\}$

The general form of the exponential function is given by: $\{(x; y) : y = ab^{x+p} + q; b > 0\}$

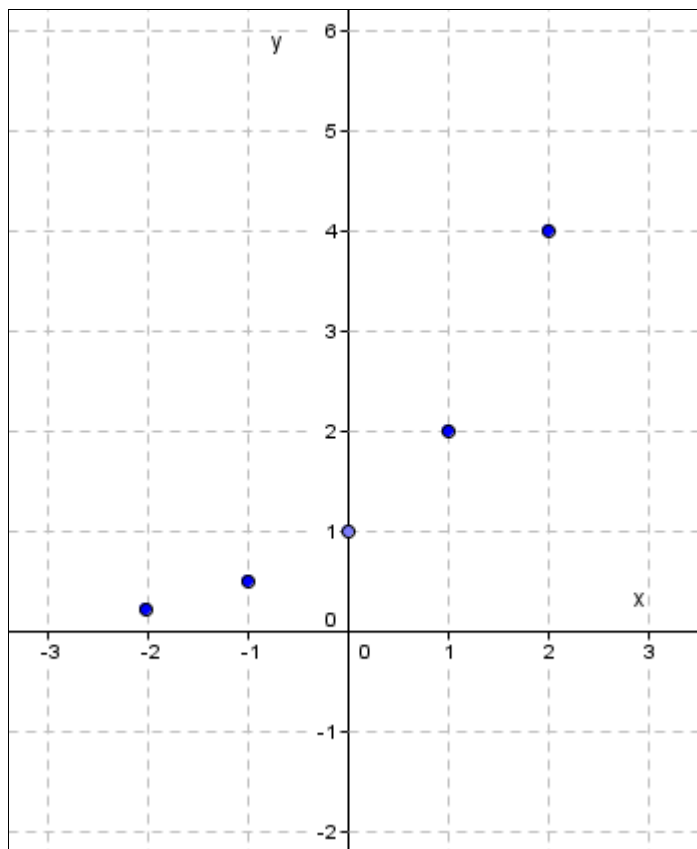
Let $p = 0$; $a = 1$; $b = 2$; $q = 0$

So we have: $\{(x; y) : y = 2^x\}$

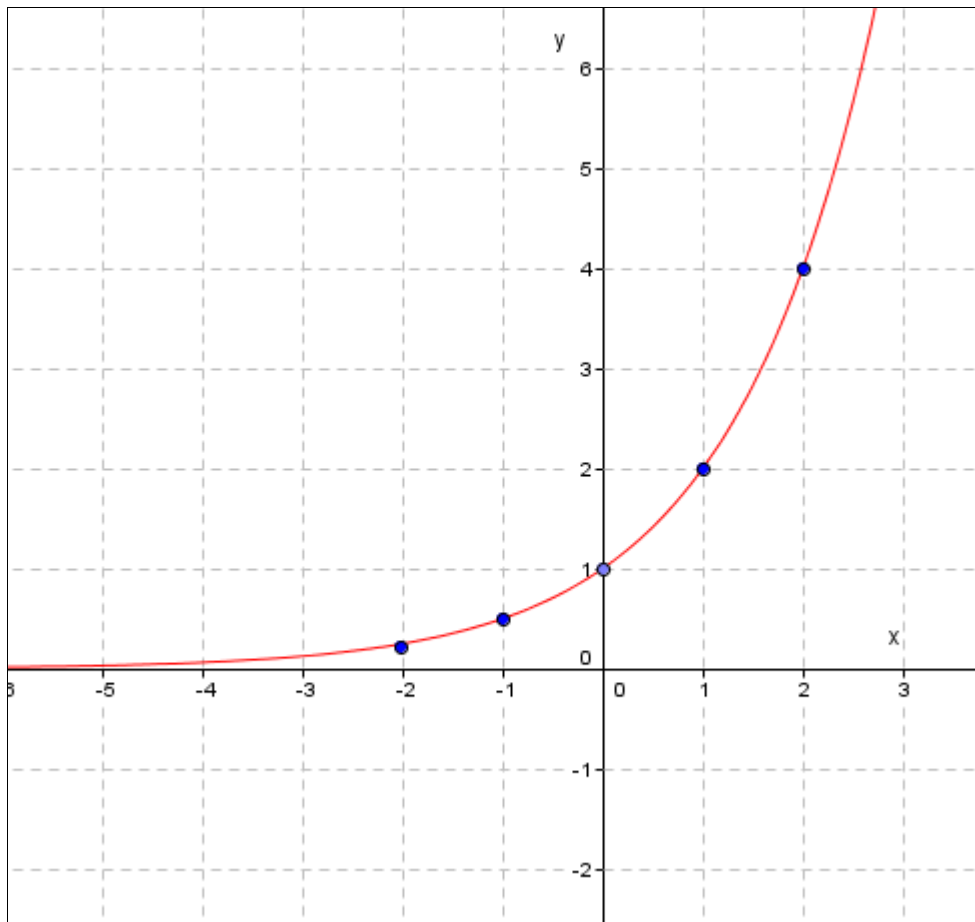
We can use point-by-point plotting to plot the graph of $y = 2^x$

x	0	1	2	-1	-2
y	1	2	4	$\frac{1}{2}$	$\frac{1}{4}$

We now plot the points in the Cartesian plane:



Now there is no restriction on the numbers in the domain as all real numbers are included. We can join the points.



Some observations:

- Domain = $\{x : x \in \mathbb{R}\}$
- Range = $\{y : y \geq 0, y \in \mathbb{R}\}$
- We note that the y-intercept is 1 and occurs at (0;1). There is no x-intercept
- The x-axis is an asymptote (horizontal)
- The graph represents a one-to-function



Worked Examples 1-2: Simple sketches of exponential functions(30 Mins)

The facilitator will now take you through a number of sketches of exponential functions.

Example 1

1.1 Sketch the graphs of:

$$y = 2^x$$

$$y = 3.2^x$$

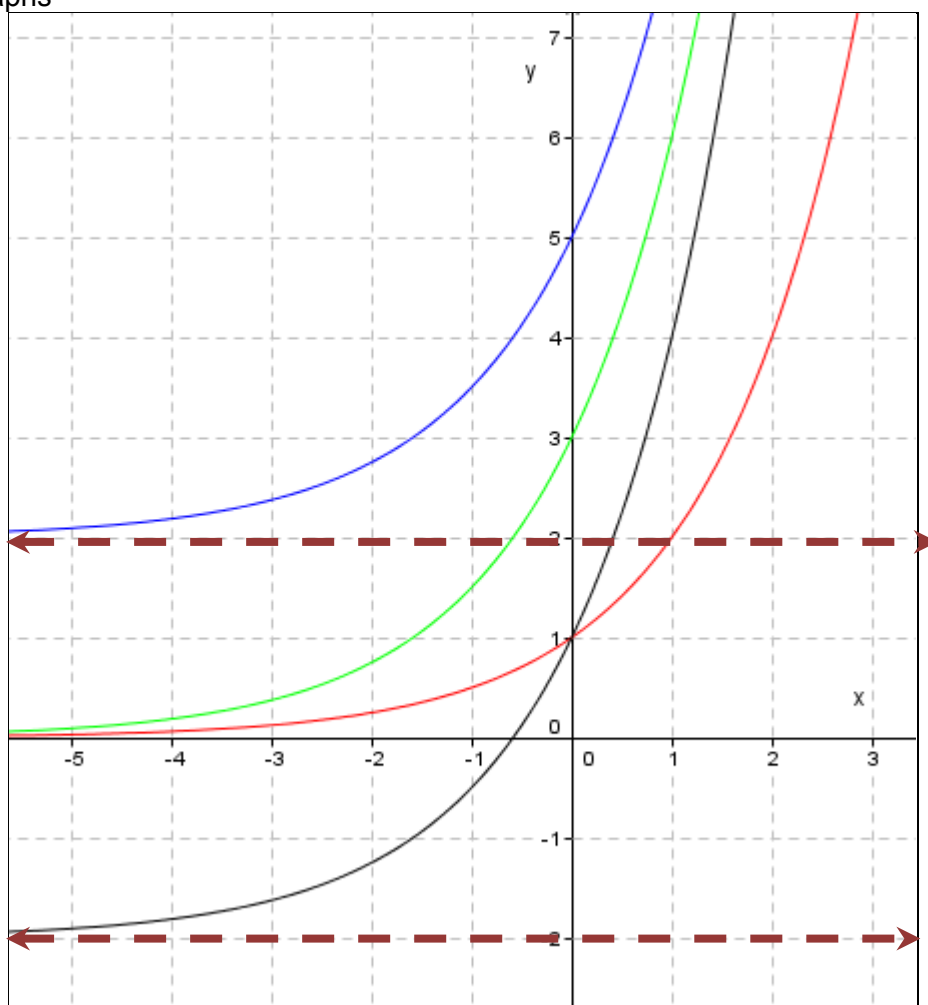
$$y = 3.2^x + 2$$

$$y = 3.2^x - 2$$

1.2 What do you observe?

Solution

1.1 Graphs



1.2 What do you observe?

$y = 2^x \rightarrow$ asymptote is negative x-axis; y intercept is $y = 1$; no x-intercept

$y = 3 \cdot 2^x \rightarrow$ asymptote is negative x-axis; y intercept is $y = 3$; no x-intercept

$y = 3 \cdot 2^x + 2 \rightarrow$ asymptote is $y = 2$; y intercept is $y = 3$; no x-intercept

$y = 3 \cdot 2^x - 2 \rightarrow$ asymptote is $y = -2$; y intercept is $y = 1$; x-intercept is -0,6 (approximately)

The asymptotes above are all horizontal asymptotes.

Please note:

$y = 2^x$ is a stretch of $y = 3 \cdot 2^x$ by a factor of 3

Example 2

2.1 Sketch the graphs of:

$$y = 3^x$$

$$y = 3^{x+1}$$

$$y = 2 \cdot 3^{x+1}$$

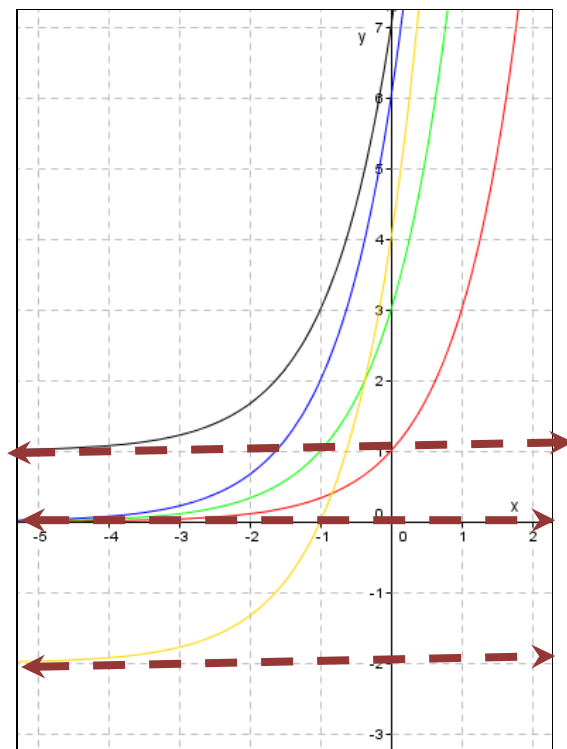
$$y = 2 \cdot 3^{x+1} + 1$$

$$y = 2 \cdot 3^{x+1} - 2$$

2.2 What do you observe?

Solution

2.1 Graphs



2.2 What do you observe?

$y = 3^x \rightarrow$ asymptote is the negative x-axis; y-intercept is $y = 1$

$y = 3^{x+1} \rightarrow$ asymptote is the negative x-axis; y-intercept is $y = 3$

$y = 2 \cdot 3^{x+1} \rightarrow$ asymptote is the negative x-axis; y-intercept is $y = 6$

$y = 2 \cdot 3^{x+1} + 1 \rightarrow$ asymptote is $y = 1$; y-intercept is $y = 7$

$y = 2 \cdot 3^{x+1} - 2 \rightarrow$ asymptote is $y = -2$; y-intercept is $y = 4$; x-intercept is $x = -4$



Activity 1-3: The exponential function

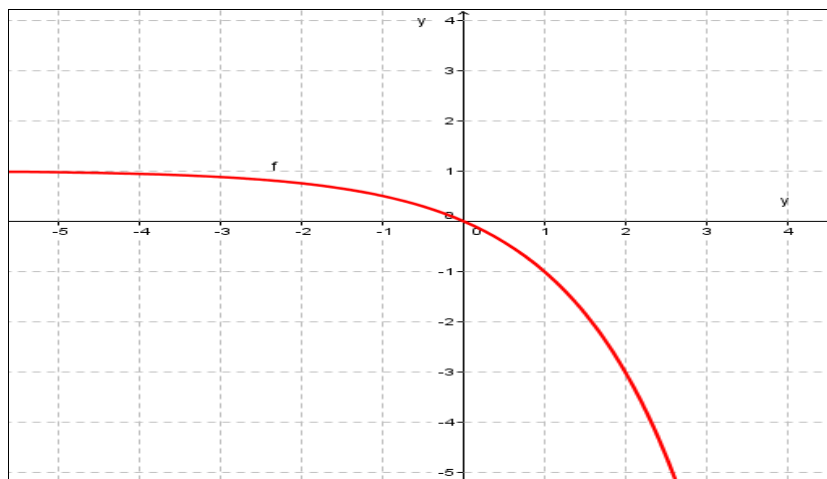
Group organisation:	Time:	Resources:	Appendix:
Groups of 6	45 min	- Flip chart - Permanent markers.	None
In your groups you will: - Select a new scribe and a spokesperson for this activity only. - Use the flipchart and permanent markers and answer the questions as per the activity.			

Activity 1

- Given $f = \{(x,y) : y = 3 \cdot 2^x - 1\}$ and $g = \{(x,y) : y = x + 2\}$
 - Sketch the graphs of f and g on the same system of axes.
 - Write down the domain of f .
 - Determine from your graphs where f and g intersect.
 - Determine the equation of h if h is formed by moving f four units upwards and two units to the right.

Activity 2

- The graph of $f(x) = 1 + a \cdot 2^x$ passes through the origin (0;0) as shown below.

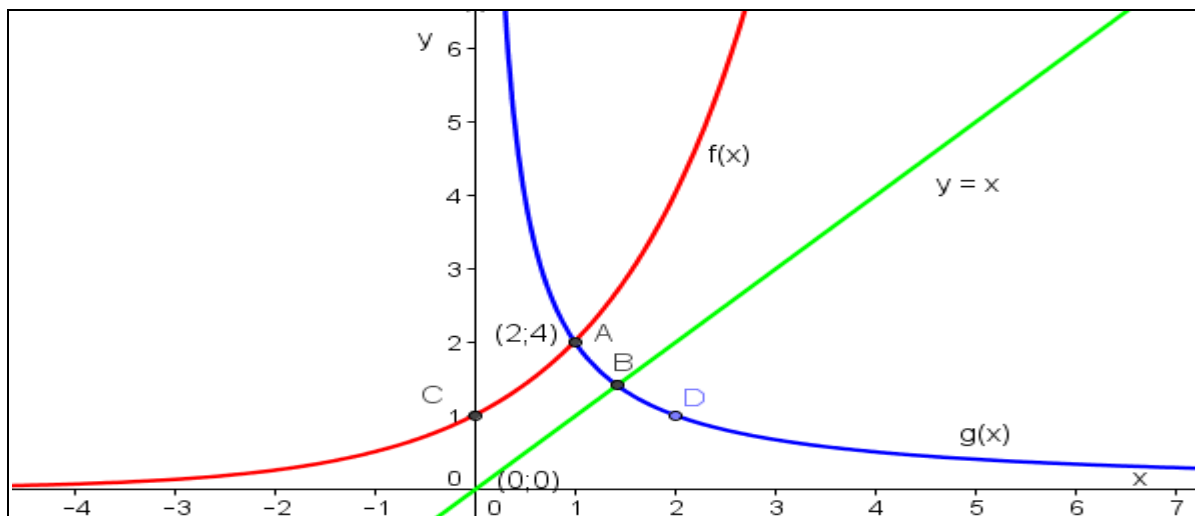


- 2.1 Show that $a = -1$.
- 2.2 Determine the value of $f(-10)$ correct to 5 decimal places.
- 2.3 Determine the value of x if $P(x;0,5)$ lies on the graph.
- 2.4 If the graph of f is shifted 2 places to the left to form h , determine the equation of f .

Activity 3

3. The graphs of

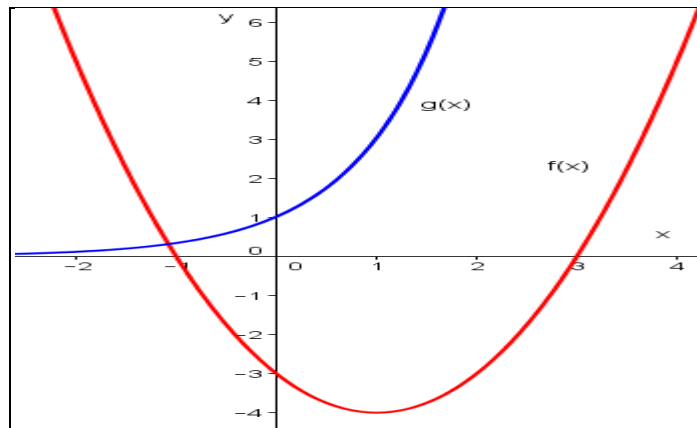
$f(x) = a^x$ and $g(x) = \frac{2}{x}; x > 0$ are represented in the diagram that follows



- 3.1 Determine the value of a in the equation $f(x) = a^x$.
- 3.2 Determine the coordinates of B, the point of intersection of g and the line $y = x$.
- 3.3 Determine the coordinates of C.
- 3.4 D is the reflection of point A in the line $y = x$. Determine the coordinates of D.
- 3.5 The graph of f is moved 3 units down to form h .
- 3.6 Determine the equation of h .
- 3.7 What will the coordinates of A become in the graph of h ?
- 3.8 What is the range of $f(x)$?

Activity 4

4. Answer the questions that follow on the graphs below:



- 4.1 Which function $f(x)$ or $g(x)$ has the form $y=3^x$.
- 4.2 What are the roots of $f(x)=0$?
- 4.3 What is the y-intercept of $g(x)$?
- 4.4 What is the range of $g(x)$?
- 4.5 Give the equation of the asymptote of $g(x)$.
- 4.6 Give the equation of the axis of symmetry of $f(x)$?

Misconceptions

Emphasise to learners the following:

- Using point-by-point plotting of exponential functions.
- Knowing that the exponential function does not cut the x-axis unless the graph is translated down the $-y$ -axis.
- How to interpret the features of drawn graphs.

Conclusion

Summarise key points of the lesson

- The features of the exponential function.
- The shape of the exponential function.

Sub-topic 6: Inverse of selected functions

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
		General concept of the inverse of a function and how the domain of the function may be restricted (in order to obtain a one-to-one function) to ensure that the inverse is a function Determine and sketch graphs of the inverses of functions defined by: $y = ax + q; y = ax^2;$ $y = b^x; (b > 0; b \neq 1)$ Focus on the following: Domain & range; intercepts on the axes; turning points; minima; maxima; shape and symmetry; average gradient; intervals on which the function increases/decreases. Revision of exponential laws and the exponential graph The definition of a logarithm The graph of the function defined by $y = \log_b x$ for cases $0 < b < 1$ and $b > 1$

Introduction

To find the inverse of functions, we interchange the x-and-y values in the given function:

This can be seen in the table below:

Function	Inverse
$y = x + 3$	$x = y + 3$
$y = 2x^2$	$x = 2y^2$
$y = \frac{6}{x}$	$x = \frac{6}{y}$
$y = 2^x$	$x = 2^y$

The inverse of linear functions

Consider:

Let $f(x) = 2x - 4$, we may write this function as $y = 2x - 4$

Thus, the inverse of f written as f^{-1} will be: $x = 2y - 4$.

Writing this inverse in the y-form we get: $y = \frac{(x + 4)}{2}$

$$\text{So } f^{-1}(x) = \frac{(x + 4)}{2}$$

Remark

We have to be able to denote the difference in notation between the function and its inverse.

The standard notation is $f(x)$ for the function and $f^{-1}(x)$ for the inverse. Do not confuse

$f^{-1}(x)$ with $\frac{1}{f(x)}$ as they do not represent the same thing.

Please note:

- Only one-to-one functions have a unique inverse.
- If the function is not one-to-one, the domain of the function must be restricted so that a portion of the graph is one-to one. You can find a unique inverse over that portion of the restricted domain.
- The domain of the function is equal to the range of the inverse. The range of the function is equal to the domain of the inverse.



Worked Example 1,2,3&4: Inverse functions-all types (60 Mins)

The facilitator will now discuss the various inverse functions.

Don't forget that any question is an important question!

Example 1

Determine the inverses of the following functions and sketch both the function and its inverse on the same system of axes. Indicate whether the given function and its inverse is a function or not, and state the domain and range of the inverse.

1.1 $y = x + 3$

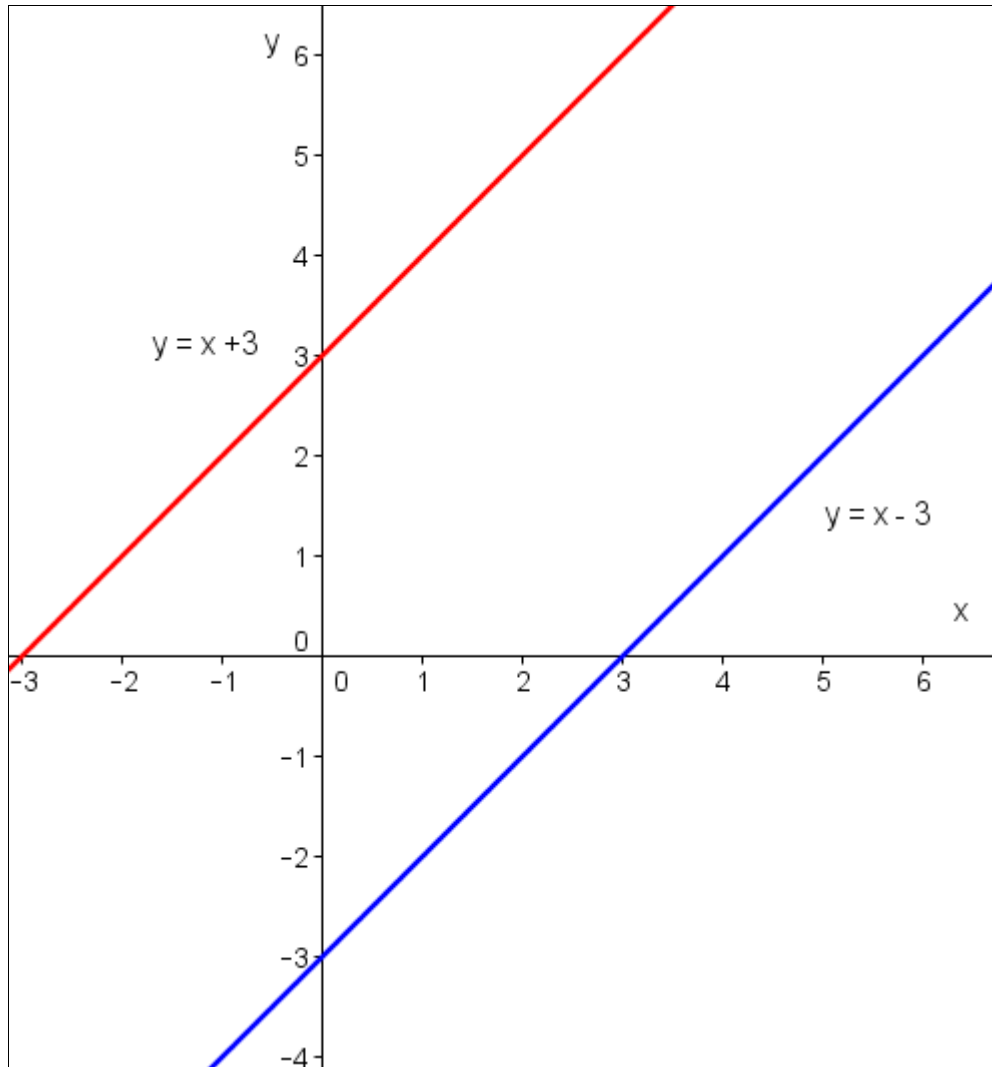
1.2 $3x + 2y = 6$

Solution

1.1

$$y = x + 3$$

The inverse is: $x = y + 3 \Rightarrow y = x - 3$ (why?)



Both $y = x + 3$ and $y = x - 3$ are one-to-one functions

In both cases:

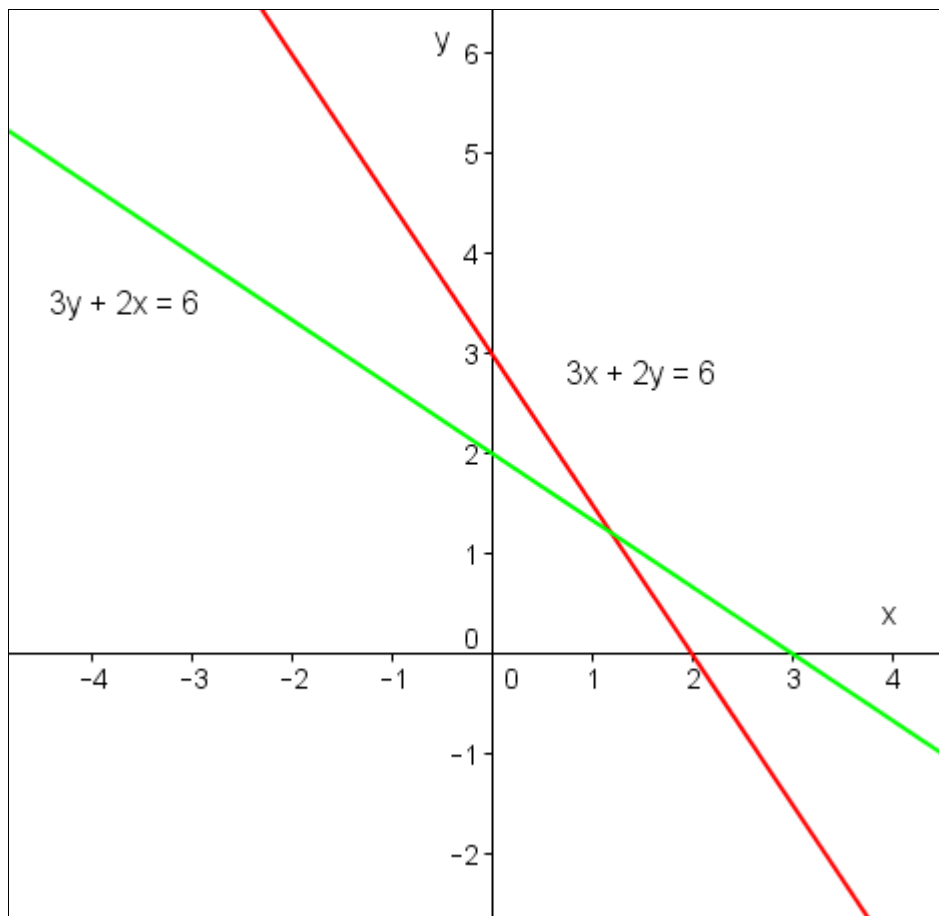
$$\text{Domain} = \{x : x \in \mathbb{R}\}$$

$$\text{Range} = \{y : y \in \mathbb{R}\}$$

1.2

$$2x + 3y = 6$$

The inverse is: $2y + 3x = 6$ (why?)



Both $3x + 2y = 6$ and $3y + 2x = 6$ are one-to-one functions

In both cases:

Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

NB: If you are given a horizontal line such as $y = 3$, its inverse will be $x = 3$. Why is this so?

Example 2

Let $f = \{(x; y): y = 9x^2\}$

$$y = 9x^2$$

Just like the linear function, interchange x and y :

Thus, the inverse is: $x = 9y^2$

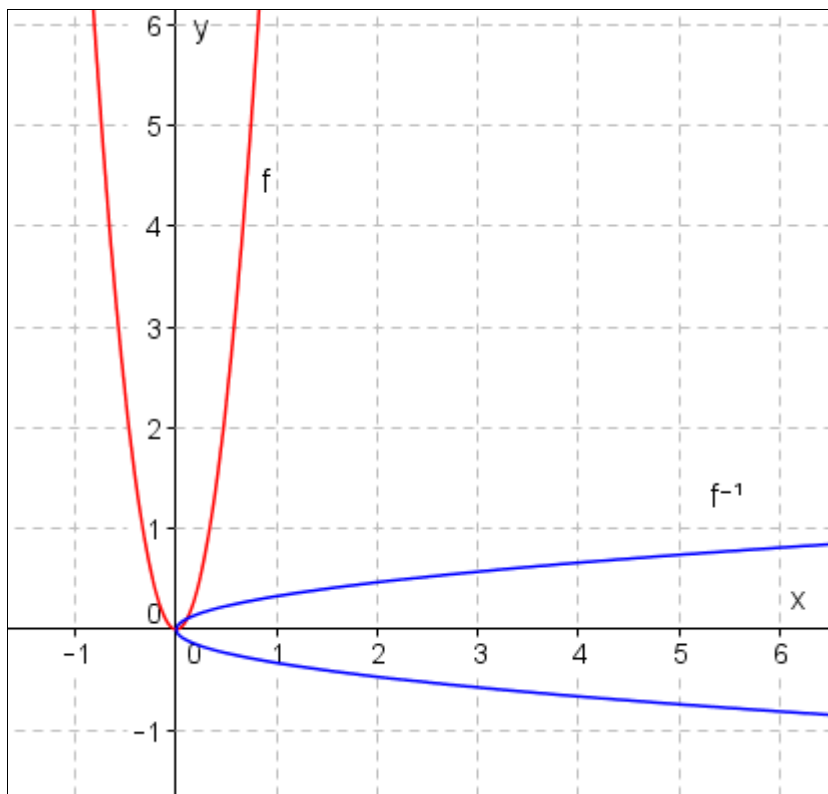
Making y the subject of the formula, we have

$$y^2 = \frac{x}{9}$$

$$\therefore \Rightarrow y = \frac{\pm\sqrt{x}}{3}$$

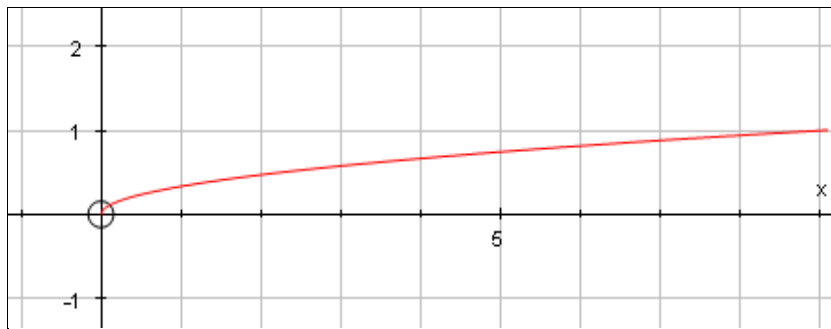
$$\text{So: } f^{-1} = \left\{ (x; y) : y = \frac{\pm\sqrt{x}}{3} \right\}$$

We may now draw both f and f^{-1} .

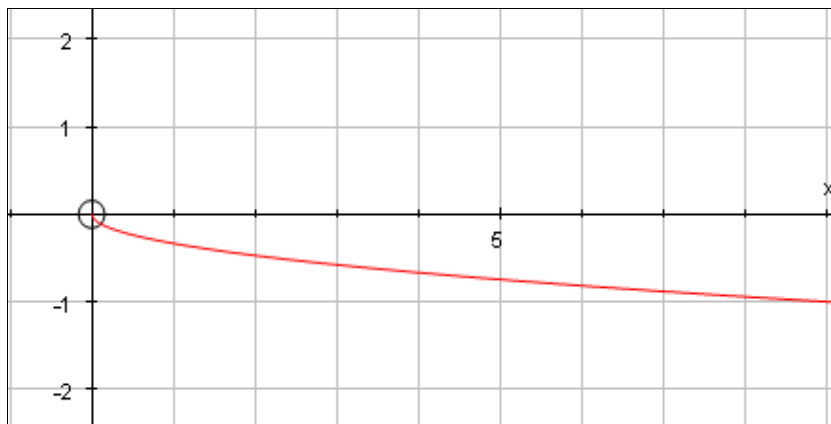


We see that this inverse is not a function (Why?). In order for the inverse to be a function we have to restrict the range, either taking positive values or negative values.

If $y \geq 0$ we have the following sketch graph, which will be a function



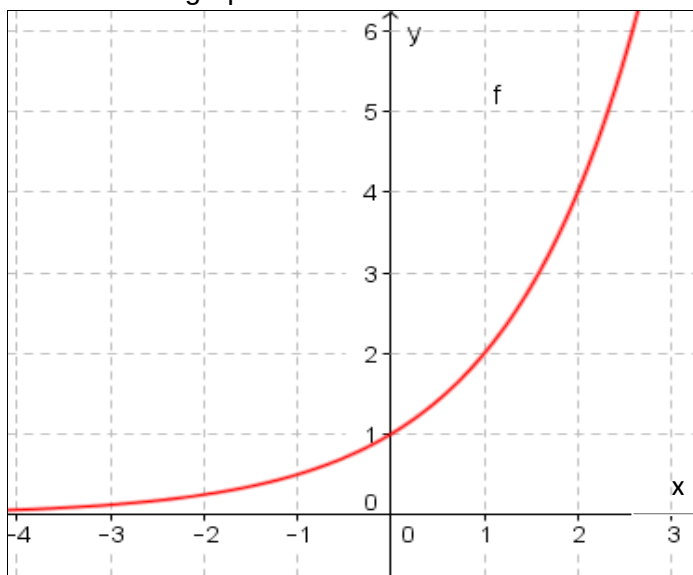
If $y \leq 0$ we have the following sketch graph, which will also be a function



Example 3

Let $f = \{(x; y) : y = 2^x\}$

We draw this graph as follows:



We see that the graph approaches the x-axis but does not cut the x-axis.

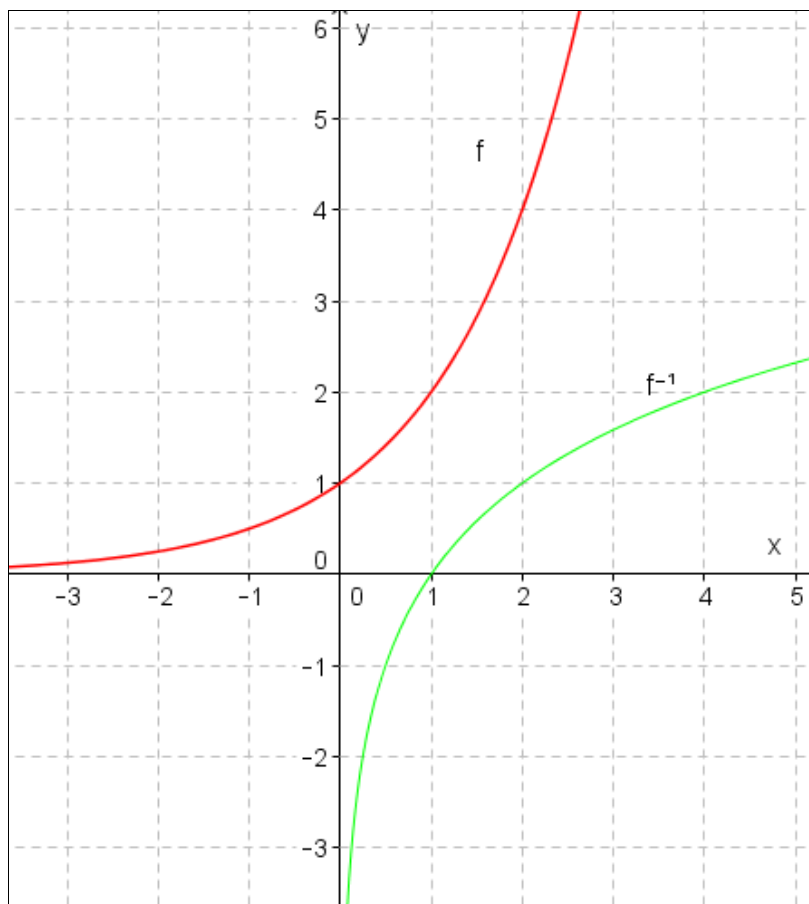
We say the graph is asymptotic at $y = 0$ (the x-axis). Although it is not possible to state what the exact minimum value of the graph is, we know that the graph will not go below the x-axis. We also see that the graph is increasing.

Now if we interchange x and y in $y = 2^x$ then the inverse is: $x = 2^y$

If we make y the subject of the formula: $y = \log_2 x$

Hence, $f^{-1} = \{(x; y) : y = \log_2 x\}$

We sketch this graph as follows:



We see that the graph cuts the x-axis at 1. The above graph gets close to the y-axis but does not cut the y-axis. We say the graph is asymptotic at $x = 0$ (the y-axis). For all $x > 0$ we see that the graph is increasing

Domain = $\{x: x > 0; x \in \mathbb{R}\}$

Range = $\{y: y \in \mathbb{R}\}$

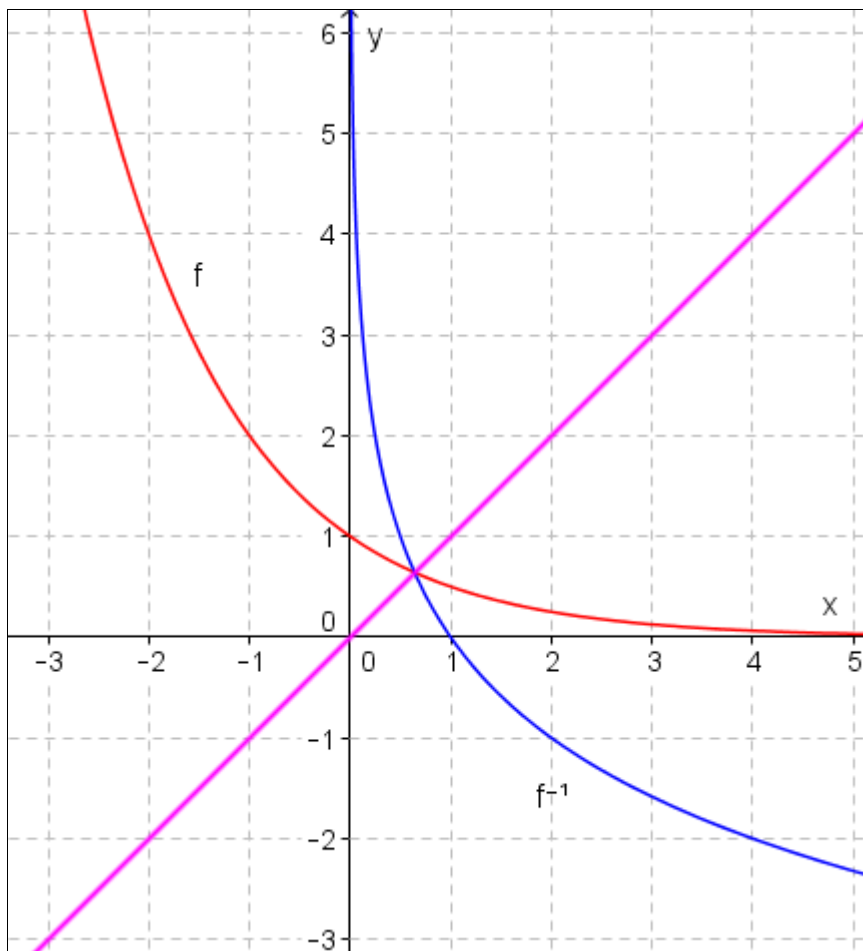
The inverse is a one-to-one function

Example 4

If $y = \left(\frac{1}{2}\right)^x$ then, using our properties of exponents, we may write this function as: $y = 2^{-x}$

The inverse of $y = 2^{-x}$ is $x = 2^{-y} \Rightarrow y = -\log_2 x$

If we draw these graphs ($y = \left(\frac{1}{2}\right)^x$ and $y = -\log_2 x$ on the same system of axes we get:



Both these graphs are decreasing.
The graphs are symmetric with respect to $y = x$.



Activity 1-3: Inverse functions

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	45 min	• Flip chart • Permanent markers.	None
In your groups you will: 1. Select a new scribe and a spokesperson for these activities. 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

Activity 1

1. Determine the inverse of each of the following functions. Sketch the inverse and determine whether the inverse is a function or not. If the inverse is not a function, restrict its range so that it becomes a function.

1.1 $y = 2x^2$

1.2 $y = -x^2$

- 1.3 Sketch the graphs of p and p^{-1} on the same system of axis where

$$p = \left\{ (x; y) : y = \frac{1}{4}x^2 \right\}$$

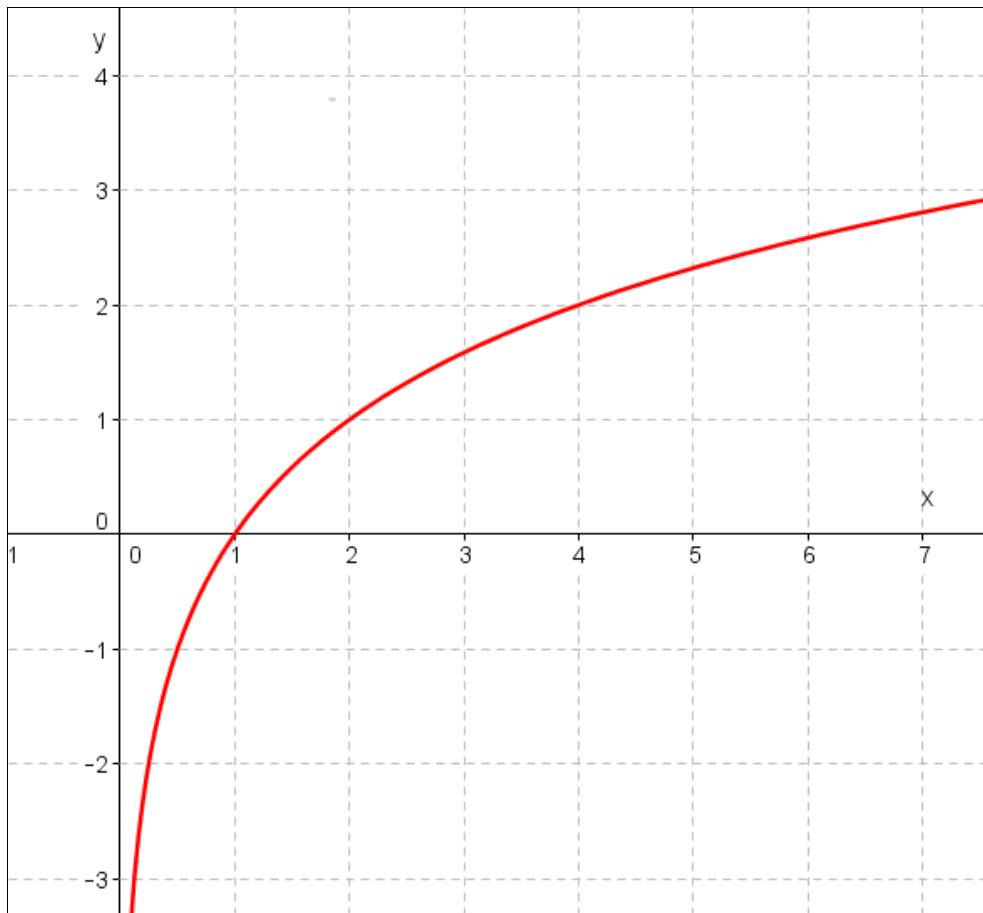
Activity 2

2. Given: $f = \{(x; y) : y = 3^x\}$
 - 2.1. Determine f^{-1}
 - 2.2. Sketch the graphs of f and f^{-1} on the same system of axes.
 - 2.3. Determine $f \cap f^{-1}$
 - 2.4. The graph of f is reflected over the x-axis to form g . Determine the equation of g .
 - 2.5. Determine the transformation from f to g .

Some key principles with respect to the logarithmic graph

Study the two graphs drawn:

(1) $f = \{(x; y) : y = \log_2 x\}$

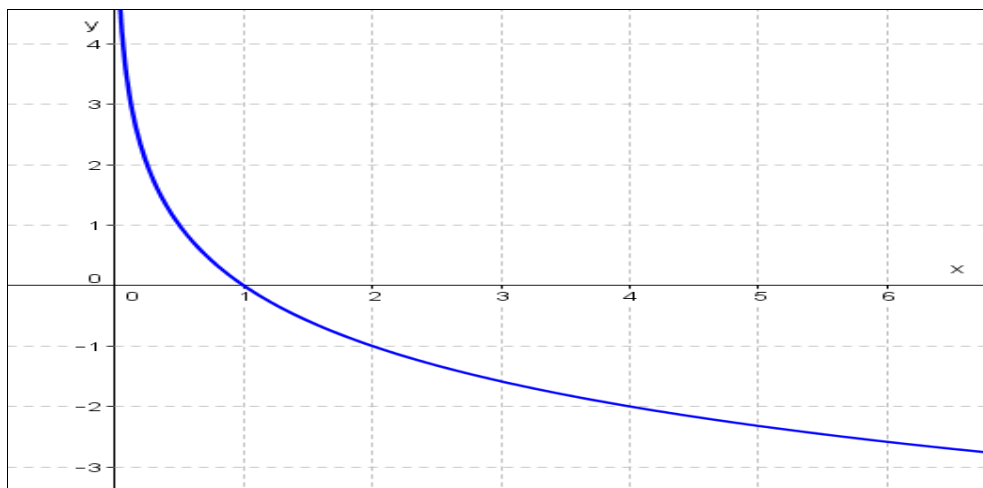


$$(2) \ g = \{(x; y) : y = \log_{\frac{1}{2}} x\}$$

$$\text{Now } y = \log_{\frac{1}{2}} x = \frac{\log x}{\log\left(\frac{1}{2}\right)} = \frac{\log x}{-\log 2} = -\log_2 x$$

We see that:

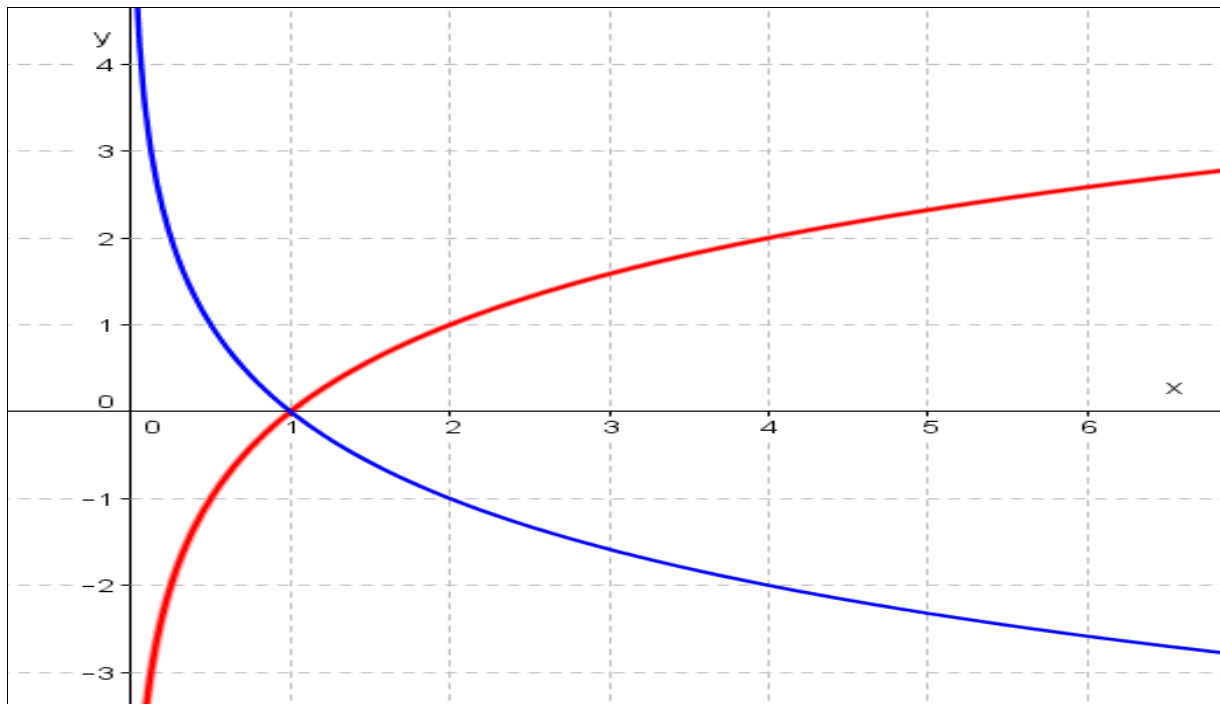
$$\log_{\frac{1}{2}} x = -\log_2 x$$



Features of the logarithmic graphs

Graph	$f = \{(x; y) : y = \log_2 x\}$	$g = \{(x; y) : y = \log_{\frac{1}{2}} x\}$
Domain	$\{x : x \in \mathbb{R}; x > 0\}$	$\{x : x \in \mathbb{R}; x > 0\}$
Range	$\{y : y \in \mathbb{R}\}$	$\{y : y \in \mathbb{R}\}$
Asymptotes	The x-axis	The x-axis
Coordinates of x-intercepts	(1;0)	(1;0)
Coordinates of y-intercepts	Nil	Nil
Increasing or decreasing function	Increasing	Decreasing

If we draw the graphs together



We note that

$y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are reflections of each other with respect to the positive x-axis.

Misconceptions

Learners should be able to determine the following features from inverse functions:

- The domain and range.
- The asymptotic behaviour of the exponential function and its inverse, the logarithmic function.
- Work with functions and inverse functions which are transformed.
- The correspondence of both the function and inverse function.

Conclusion

Summarise key points of the inverse functions:

- How to determine the inverse of a given function.
- How to modify the domain of the inverse so the inverse is a function.
- Key steps in drawing the inverse functions.

Sub-topic 7: Trigonometric functions

CAPS extraction indicating progression from grades 10-12

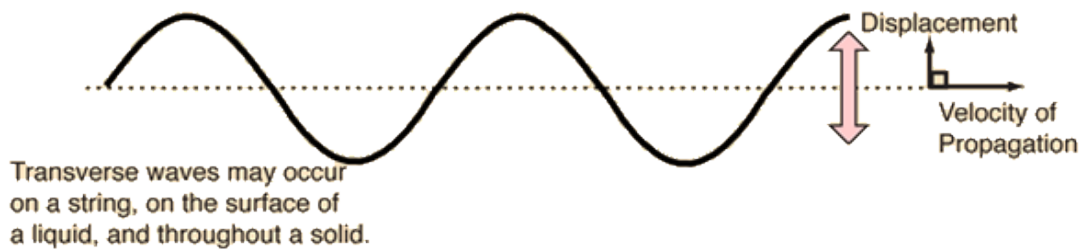
Grade 10	Grade 11	Grade 12
Point by point plotting of basic graphs defined by $y = \sin x; y = \cos x$ and $y = \tan x$ for $x \in [0^\circ; 360^\circ]$ Study the effect of a and q on the graphs defined by: $y = a \sin x + q; y = a \cos x + q$ and $y = a \tan x + q$ where $a, q \in \mathbb{R}$ for $x \in [0^\circ; 360^\circ]$ Sketch graphs, find the equations of given graphs and interpret graphs	Point by point plotting of basic graphs defined by $y = \sin x; y = \cos x$ and $y = \tan x$ for $x \in [0^\circ; 360^\circ]$ Investigate the effect of the parameter p on the graphs defined by $y = \sin(x + p); y = \cos(x + p)$ and $y = \tan(x + p)$ Draw sketch graphs defined by $y = a \sin k(x + p); y = a \cos k(x + p)$ and $y = a \tan k(x + p)$ with at most two parameters at a time.	

Introduction

When introducing trigonometric functions it is important to start with what learners may know or have experienced.

Learners may be asked about waves at the beach or water waves which may be observed in a ripple tank. These water waves are examples of transverse waves. Transverse waves are waves in which the vibrations of the particles or medium in which the wave travels are at right angles to the direction in which the wave travels.

For transverse waves the displacement of the medium is perpendicular to the direction of propagation of the wave. A ripple on a pond and a wave on a string are easily visualised as transverse waves.

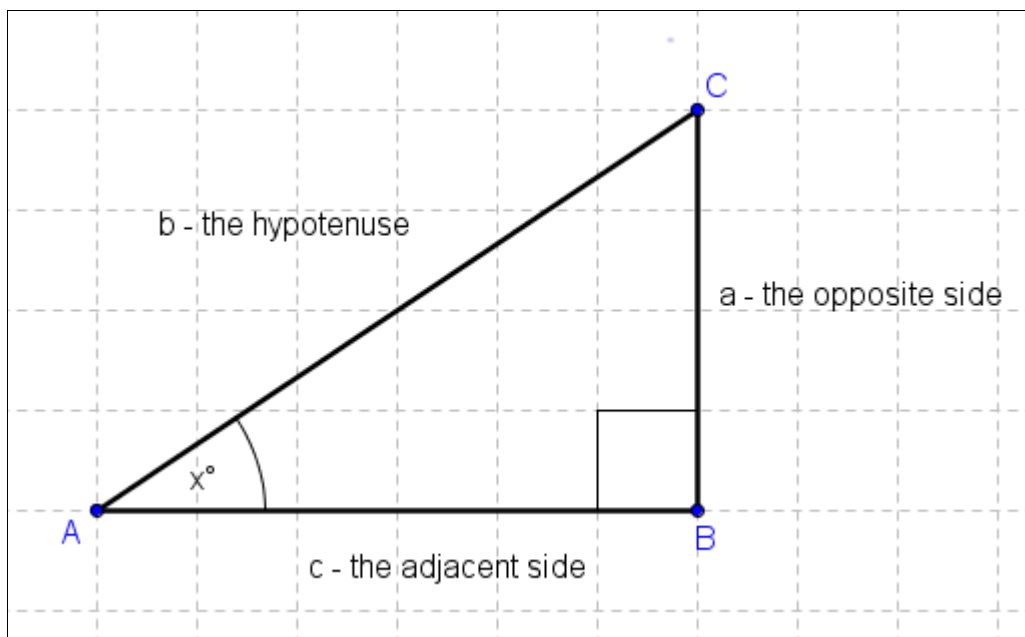


Transverse waves cannot propagate in a gas or a liquid because there is no mechanism for driving motion perpendicular to the propagation of the wave.

(from <http://hyperphysics.phy-astr.gsu.edu/hbase/sound/tralon.html>)

The most basic trigonometric graph

Learners should recall the following fundamental identities:



$$\sin x = \frac{a}{b}$$

$$\cos x = \frac{c}{b}$$

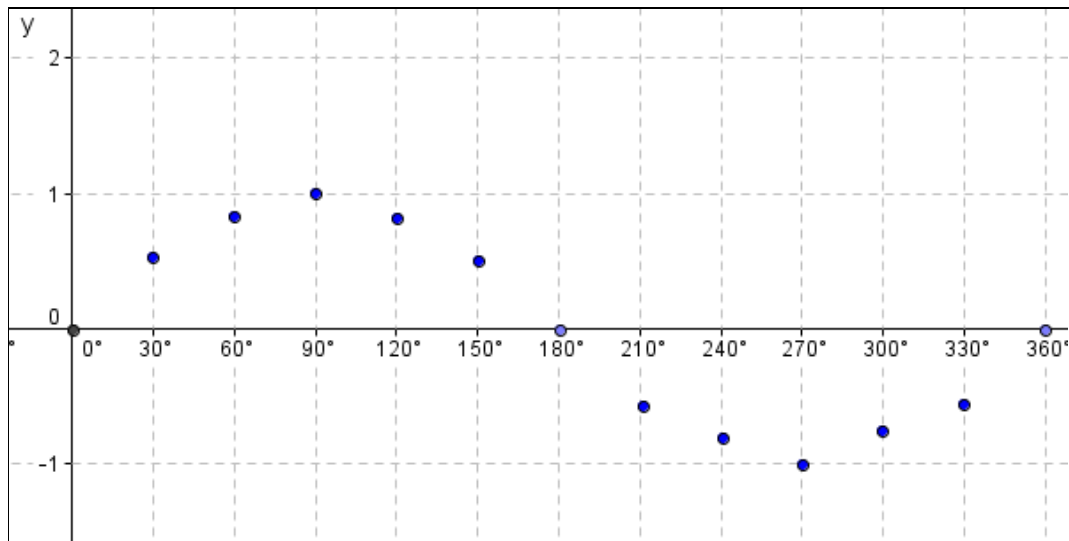
$$\tan x = \frac{a}{c}$$

From these we have the most basic trigonometry graph: $y = \sin x$

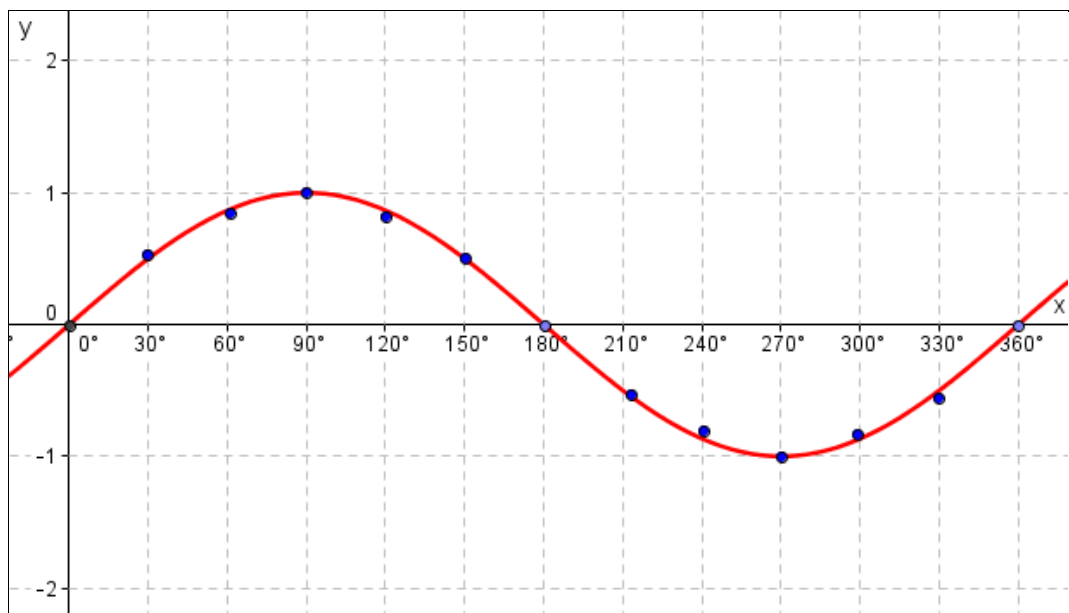
We can select a few values using the calculator to sketch this graph:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin x	0	0,5	0,866	1	0,866	0,5	0	-0,5	-0,866	-1	-0,866	-0,5	0

We can select a few points to draw this graph for $[0^\circ; 360^\circ]$



We can now join these points:

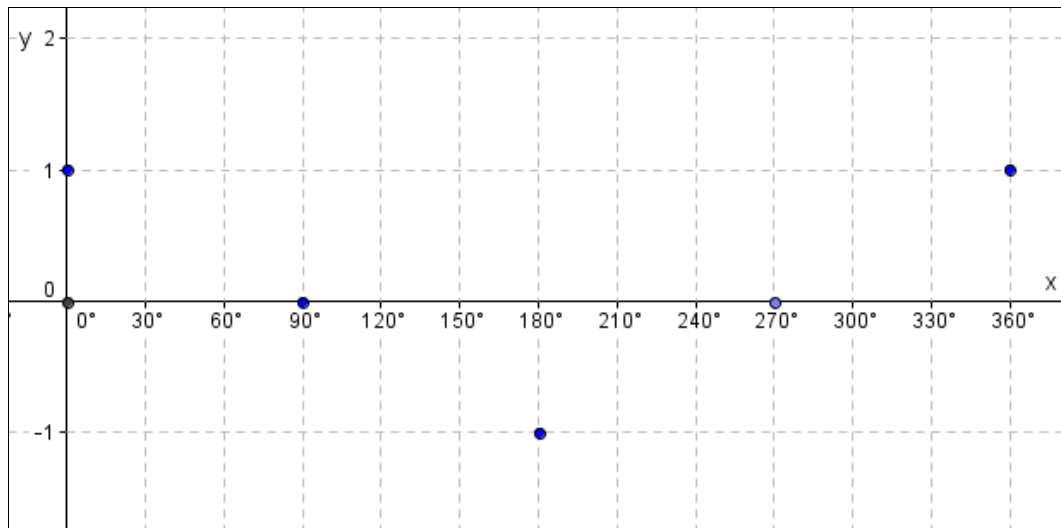


We can do the same for $y = \cos x$

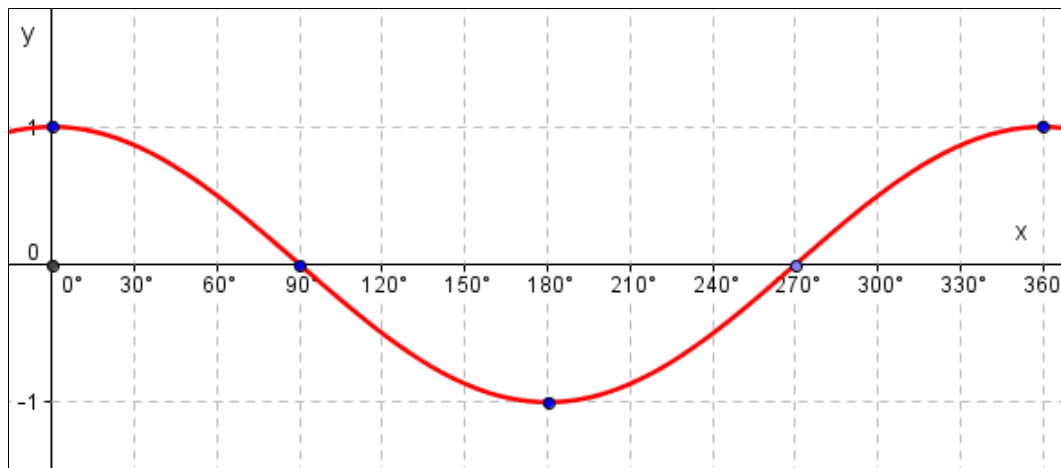
Draw your own table of values:

We will select a few:

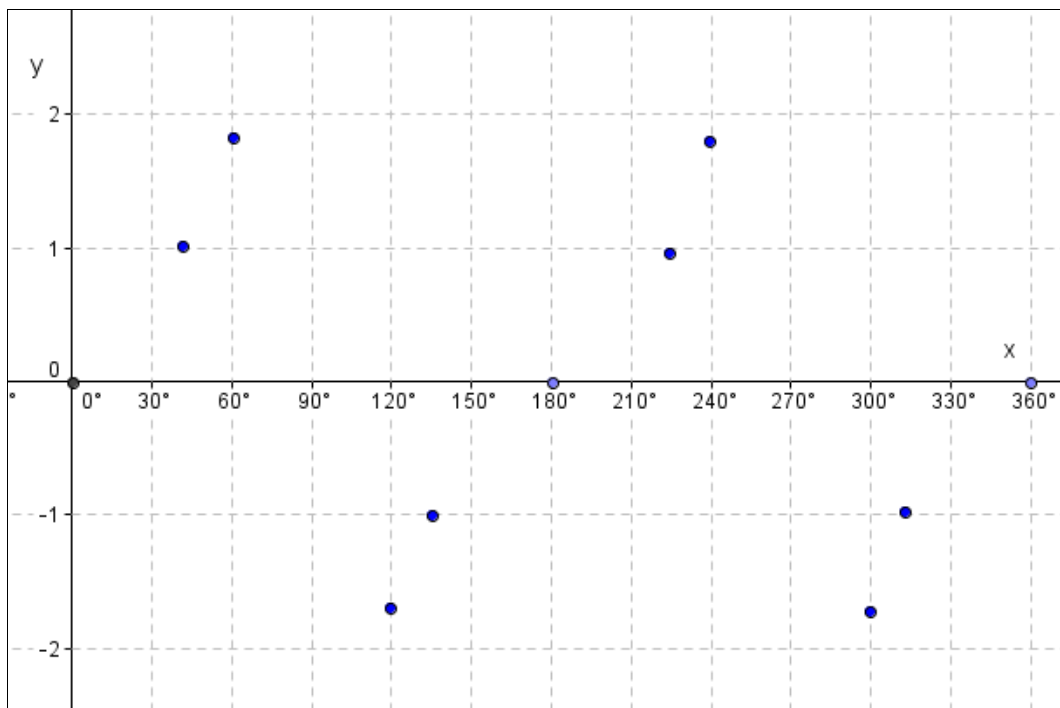
x	0°	90°	180°	270°	360°
cos x	1	0	-1	0	1



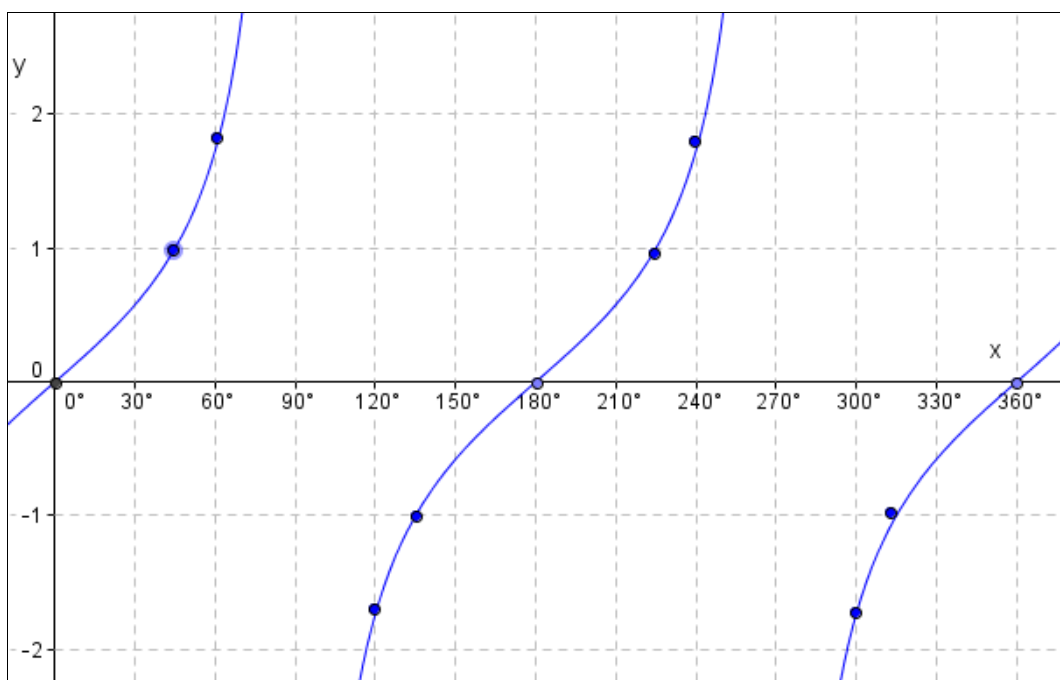
We can join the points:



We can do the same for $y = \tan x$



We join the points.



Some basic properties

Property	$y = \sin x$	$y = \cos x$	$y = \tan x$
Shape	Smooth wave-like curve	Smooth wave-like curve	A smooth curve which is discontinuous at 90° and 270°
Domain	$\{x : 0^\circ \leq x \leq 360^\circ\}$	$\{x : 0^\circ \leq x \leq 360^\circ\}$	$\{x : 0^\circ \leq x \leq 360^\circ; x \neq 90^\circ \text{ or } x \neq 270^\circ\}$
Range	$\{y : -1 \leq y \leq 1\}$	$\{y : -1 \leq y \leq 1\}$	$\{y : -\infty \leq y \leq \infty\}$
Zeros	$x = 0^\circ$ or 180° or 360°	$x = 90^\circ$ or 270°	$x = 0^\circ$ or 180° or 360°
Maximum	1	1	∞ (infinity)
Minimum	-1	-1	$-\infty$ (negative infinity)



Worked Example 1&2: Graphs of different trigonometric Functions (10 Mins)

The facilitator will now discuss these functions with you.

Don't forget that any question is an important question!

Example 1

1.1 Sketch the following graphs on the same system of axis.

$$y = \sin x$$

$$y = \sin x + 1$$

$$y = 2 \sin x$$

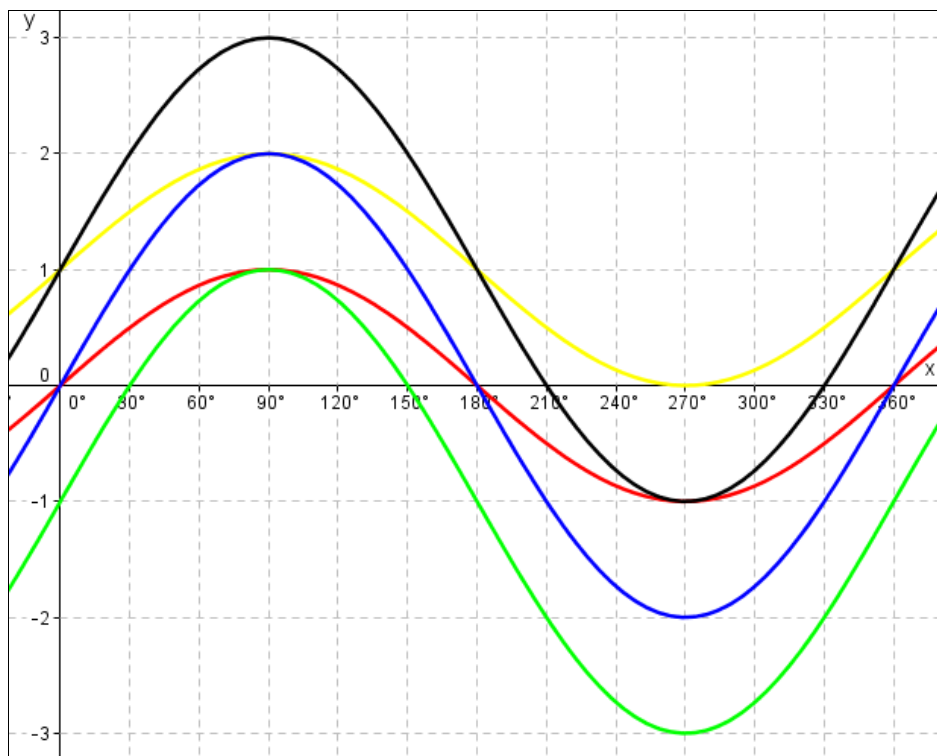
$$y = 2 \sin x + 1$$

$$y = 2 \sin x - 1$$

1.2 What do you notice?

Solution

1.1 Graph



Try to match the above equations with the graphs drawn.

1.2 We notice that the period of the graph is the same in each case; that of 360° .

This means that the “shape” of the graph will repeat after 360° . The amplitude is the maximum displacement from the x-axis; the zeros are the values for which the graph cuts the x-axis; the range is the set of y-values for which the graph is defined. A summary of these properties appears in the table below:

Function	<i>Amplitude</i>	<i>Period</i>	<i>Zeros</i>	<i>Range</i>
$y = \sin x$	1	360°	$0^\circ; 180^\circ; 360^\circ$	$[-1; 1]$
$y = \sin x + 1$	2	360°	$90^\circ; 270^\circ$	$[0; 1]$
$y = 2 \sin x$	2	360°	$0^\circ; 180^\circ; 360^\circ$	$[-2; 2]$
$y = 2 \sin x + 1$	3	360°	$210^\circ; 330^\circ$	$[-1; 3]$
$y = 2 \sin x - 1$	3	360°	$-330^\circ; -210^\circ;$ $30^\circ; 150^\circ$	$[1; -3]$

Note: the range $[-1;1]$ can be written as $-1 \leq y \leq 1$

$y = \sin x + 1 \rightarrow$ graph of $y = \sin x$ "moved" or translated 1 unit upward along the y-axis

$y = 2 \sin x - 1 \rightarrow$ graph of $y = \sin x$ stretched by a factor of 2 then translated down 1 unit down the y-axis

Graphs of the type $\{(x;y) : y = a \cos x + q\}$

Example 2

Sketch the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

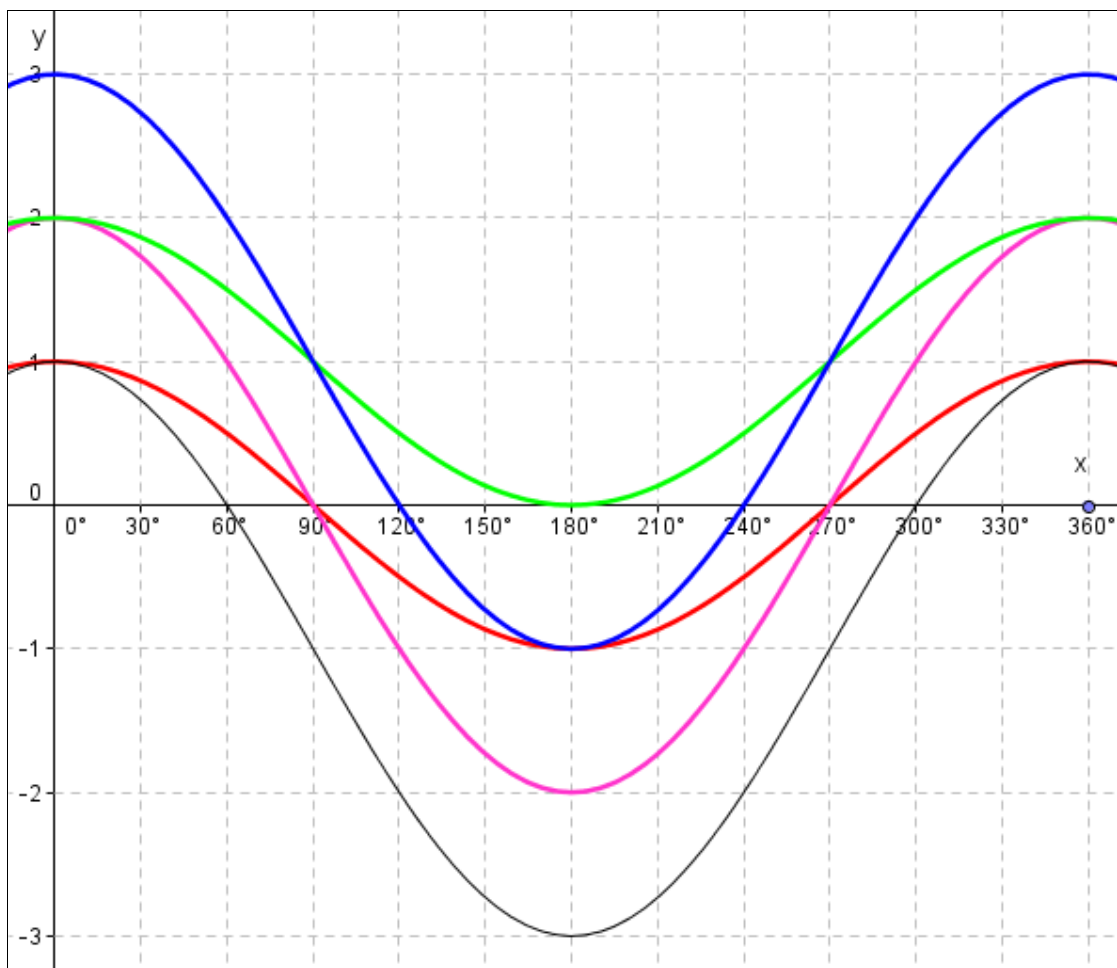
$$y = \cos x$$

$$y = \cos x + 1$$

$$y = 2 \cos x$$

$$y = 2 \cos x + 1$$

$$y = 2 \cos x - 1$$



Please note the following:

The period above 360° is very useful when finding the general solutions of trigonometric functions of sine and cosine.

It means that the angle (solution) is repeated after 360° .

If we were to find the general solution of the equation

$$\sin x = \frac{1}{2}$$

Then we have:

$$\sin x = \frac{1}{2}$$

$\Rightarrow x = 30^\circ$ or 150° (specific solutions)

and the general solutions are:

$$x = 30^\circ \pm k.360^\circ \text{ or } 150^\circ \pm k.360^\circ \quad k \in \{0;1;2;3;\dots\}$$



Worked Example 3: Graphs of the type:

$$\{(x;y) : y = a \tan x + q\}$$

Functions (5 Mins)

The facilitator will now discuss these functions with you.

Don't forget that any question is an important question!

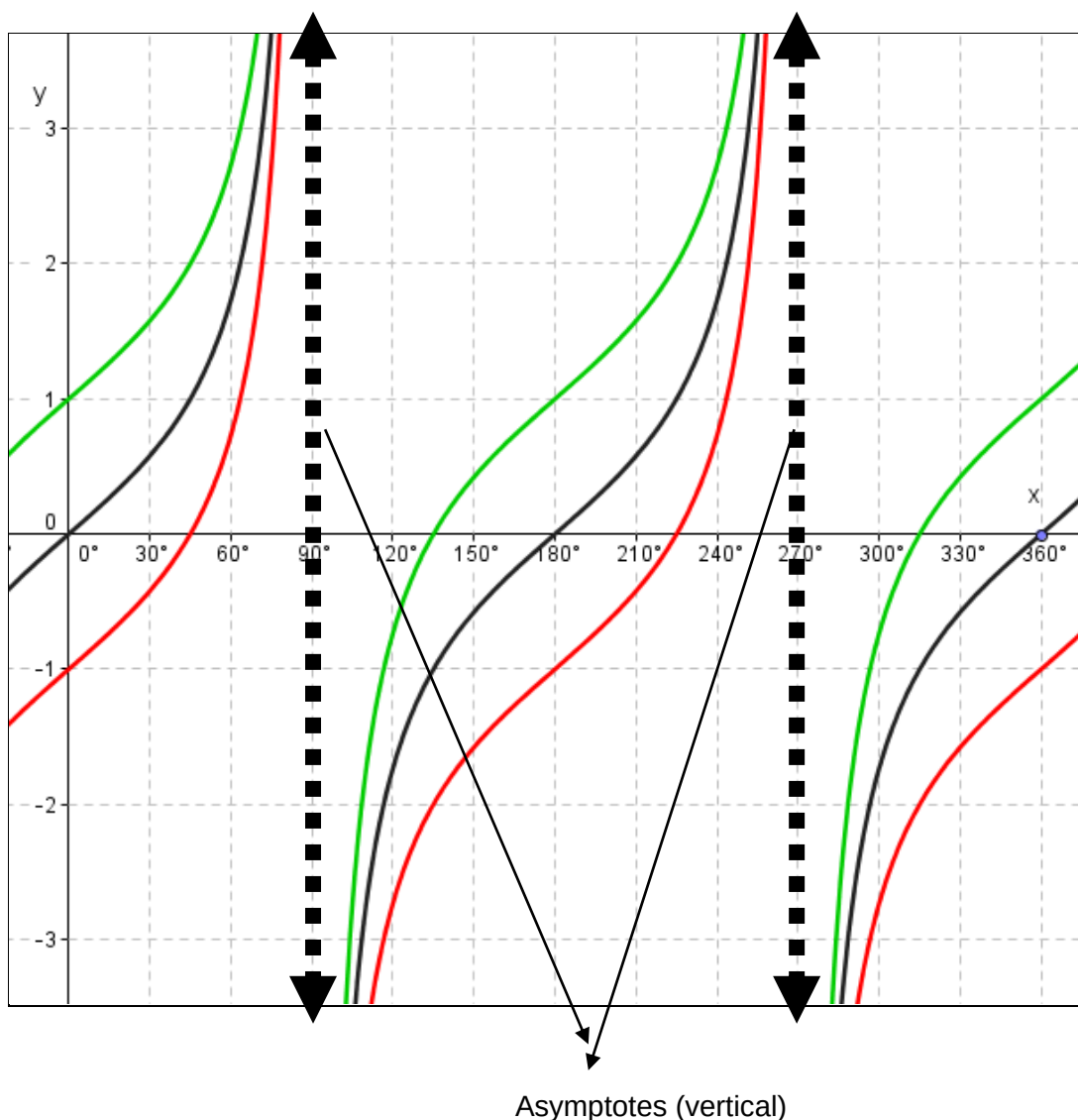
Example 3

Sketch the following graphs on the same system of axis.

$$y = \tan x$$

$$y = \tan x + 1$$

$$y = \tan x - 1$$



Function	Amplitude	Period	Zeros	Range	Asymptotes
$y = \tan x$	None	180°	$0^\circ; 180^\circ; 360^\circ$	$(-\infty; \infty)$	$90^\circ; 270^\circ$
$y = \tan x + 1$	None	180°	$135^\circ; 315^\circ$	$(-\infty; \infty)$	$90^\circ; 270^\circ$
$y = \tan x - 1$	None	180°	$45^\circ; 225^\circ$	$(-\infty; \infty)$	$90^\circ; 270^\circ$

Note :

$y = \tan x + 1 \rightarrow$ graph of $y = \tan x$ "moved" up 1 unit

$y = \tan x - 1 \rightarrow$ graph of $y = \tan x$ "moved" down 1 unit

Please note the following:

The period (180°) of the graph of $y = \tan x$ is very useful to find general solutions of trigonometric function involving $y = \tan x$.

It means that the angle (solution) is repeated after 180° .

For example: Determine the general solution of $\tan x = -\sqrt{3}$

$$\tan x = -\sqrt{3}$$

$\Rightarrow x = 120^\circ$ or 300° (specific solutions)

and the general is:

$$x = 120^\circ \pm k.180^\circ \quad k \in \{0;1;2;\dots\}$$

NB: 300° is included in the solution (when $k = 1$)

 Activity 2: Interpretation of graphs			
Group organisation:	Time:	Resources:	Appendix:
Individual	10 min	Participants Handout	None
You will complete the table as explained by the facilitator.			

Activity 2

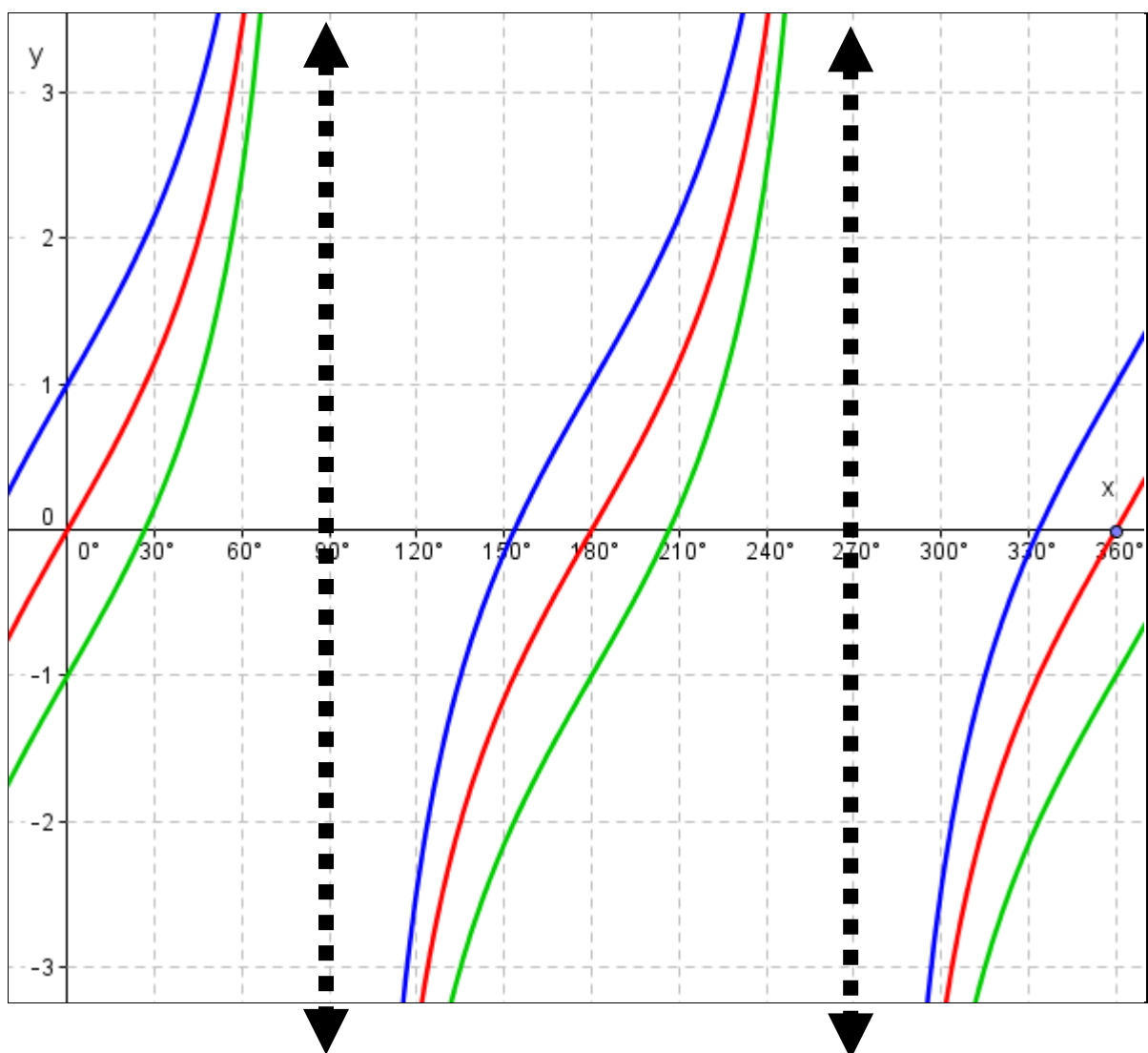
In this activity, graphs are drawn for you. You must interpret these drawings and complete the table in part 2 of each question.

The graphs of the following functions are drawn below:

$$y = 2 \tan x$$

$$y = 2 \tan x + 1$$

$$y = 2 \tan x - 1$$



Complete the following table:

Function	<i>Amplitude</i>	<i>Period</i>	<i>Zeros</i>	<i>Range</i>	<i>Asymptotes</i>
$y = 2 \tan x$					
$y = 2 \tan x + 1$					
$y = 2 \tan x - 1$					



Worked Examples: Various types Functions (40 Mins)

The facilitator will now discuss these functions with you.

Don't forget that any question is an important question!

Graphs of the type $\{(x;y) : y = \sin(kx)\}$

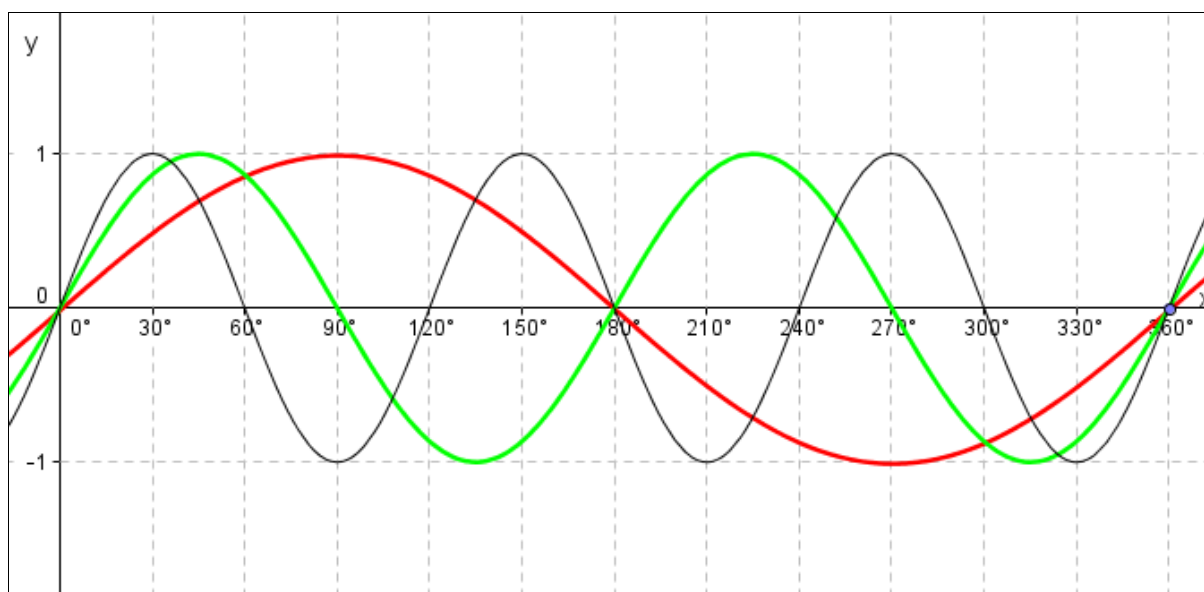
Example 4

Draw the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \sin x$$

$$y = \sin 2x$$

$$y = \sin 3x$$



Completed table:

Function	<i>Amplitude</i>	<i>Period</i>	<i>Zeros</i>	<i>Range</i>
$y = \sin x$	1	360°	$0^\circ; 180^\circ; 360^\circ$	$[-1; 1]$
$y = \sin 2x$	1	$\frac{360^\circ}{2} = 180^\circ$	$0^\circ; 90^\circ; 180^\circ; 270^\circ; 360^\circ$	$[-1; 1]$
$y = \sin 3x$	1	$\frac{360^\circ}{3} = 120^\circ$	$0^\circ; 60^\circ; 120^\circ; 180^\circ; 240^\circ; 300^\circ; 360^\circ$	$[-1; 1]$

Graphs of the type $\{(x; y) : y = \cos(kx)\}$

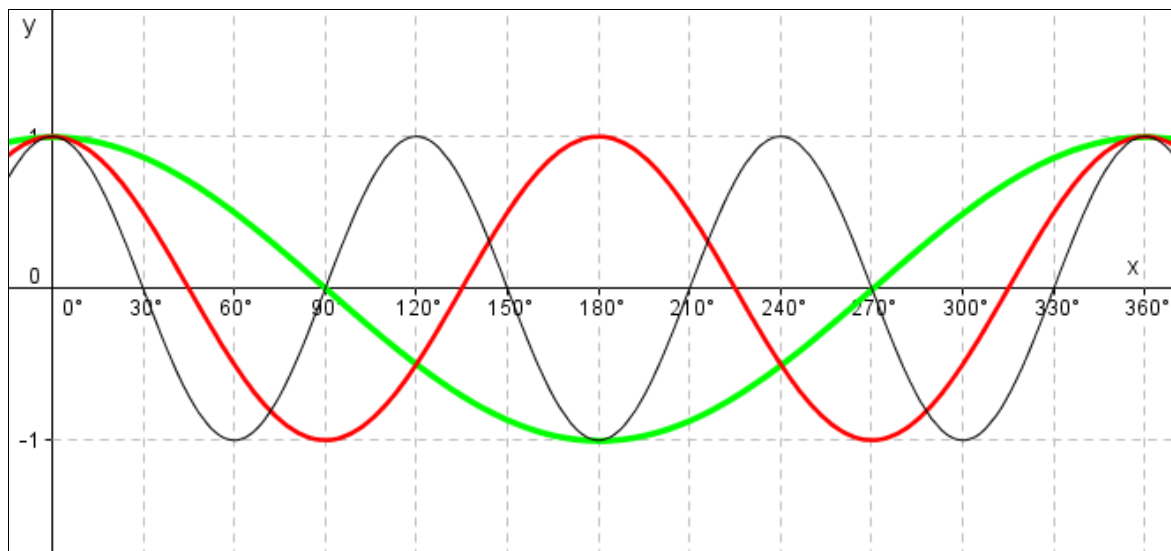
Example 5

Draw the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \cos x$$

$$y = \cos 2x$$

$$y = \cos 3x$$



Complete table:

Function	<i>Amplitude</i>	<i>Period</i>	<i>Zeros</i>	<i>Range</i>
$y = \cos x$	1	360°	$90^\circ; 270^\circ$	$[-1;1]$
$y = \cos 2x$	1	360°	$45^\circ; 135^\circ;$ $225^\circ; 315^\circ$	$[-1;1]$
$y = \cos 3x$	1	360°	$30^\circ; 90^\circ;$ $150^\circ; 210^\circ;$ $270^\circ; 330^\circ$	$[-1;1]$

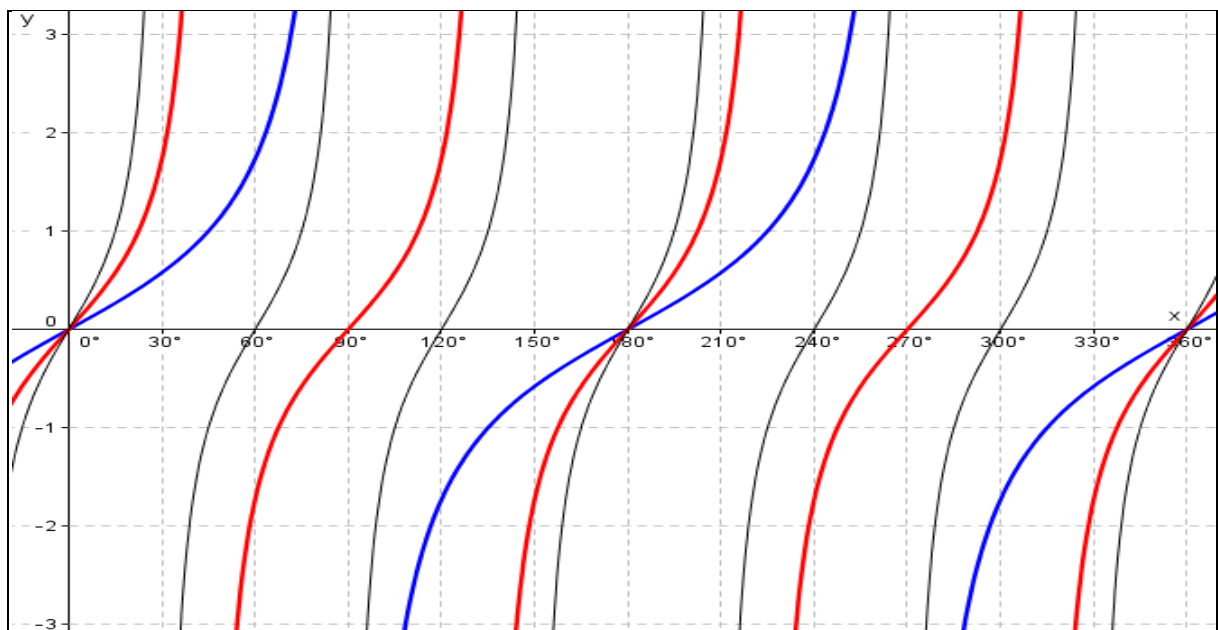
Graphs of the type $\{(x;y) : y = \tan(kx)\}$

Draw the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \tan x$$

$$y = \tan 2x$$

$$y = \tan 3x$$



Completed table:

Function	Amplitude	Period	Zeros	Range	Asymptotes
$y = \tan x$	–	180°	$0^\circ; 180^\circ; 360^\circ$	$[-\infty; \infty]$	$90^\circ; 270^\circ$
$y = \tan 2x$	–	90°	$0^\circ; 90^\circ; 180^\circ; 270^\circ; 360^\circ$	$[-\infty; \infty]$	$45^\circ; 135^\circ; 225^\circ; 315^\circ$
$y = \tan 3x$	–	60°	$0^\circ; 60^\circ; 120^\circ; 180^\circ; 240^\circ; 300^\circ; 360^\circ$	$[-\infty; \infty]$	$30^\circ; 90^\circ; 150^\circ; 210^\circ; 270^\circ; 330^\circ$

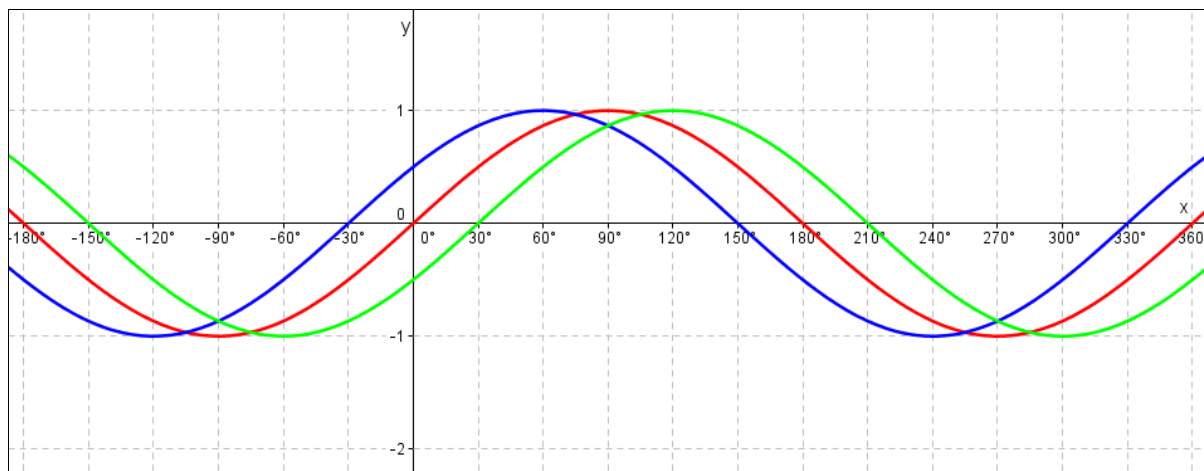
Graphs of the type: $\{(x;y): y=\sin(x+p)\}$

Draw the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \sin x$$

$$y = \sin(x + 30^\circ)$$

$$y = \sin(x - 30^\circ)$$



Completed table:

Function	Amplitude	Period	Zeros	Range
$y = \sin x$	1	360°	$0^\circ; 180^\circ; 360^\circ$	$[-1; 1]$
$y = \sin(x + 30^\circ)$	1	360°	$-30^\circ; 150^\circ; 330^\circ$	$[-1; 1]$
$y = \sin(x - 30^\circ)$	1	360°	$30^\circ; 210^\circ$	$[-1; 1]$

Note :

$y = \sin(x - 30^\circ) \rightarrow$ graph of $y = \sin x$ translated or "moved" 30° to the right (on the x-axis)

$y = \sin(x + 30^\circ) \rightarrow$ graph of $y = \sin x$ translated or "moved" 30° to the left (on the x-axis)

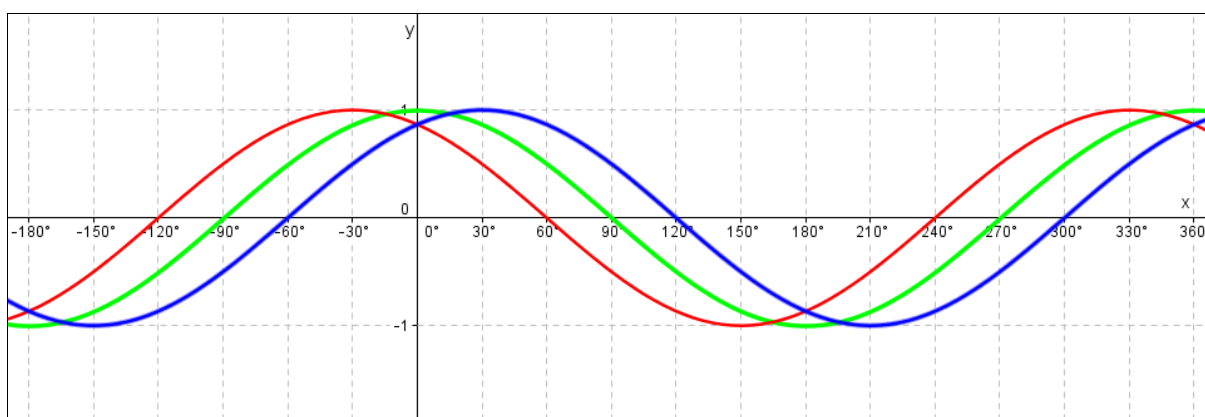
Graphs of the type $\{(x,y) : y = \cos(x + p)\}$

Draw the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \cos x$$

$$y = \cos(x + 30^\circ)$$

$$y = \cos(x - 30^\circ)$$



Completed table:

Function	Amplitude	Period	Zeros	Range
$y = \cos x$	1	360°	$-90^\circ; 90^\circ; 270^\circ$	$[-1; 1]$
$y = \cos(x + 30^\circ)$	1	360°	$-60^\circ; 120^\circ; 300^\circ$	$[-1; 1]$
$y = \cos(x - 30^\circ)$	1	360°	$-120^\circ; 60^\circ; 240^\circ$	$[-1; 1]$

Note :

$y = \cos(x - 30^\circ) \rightarrow$ graph of $y = \cos x$ translated 30° to the right on the x-axis

$y = \cos(x + 30^\circ) \rightarrow$ graph of $y = \cos x$ translated 30° to the left on the x-axis

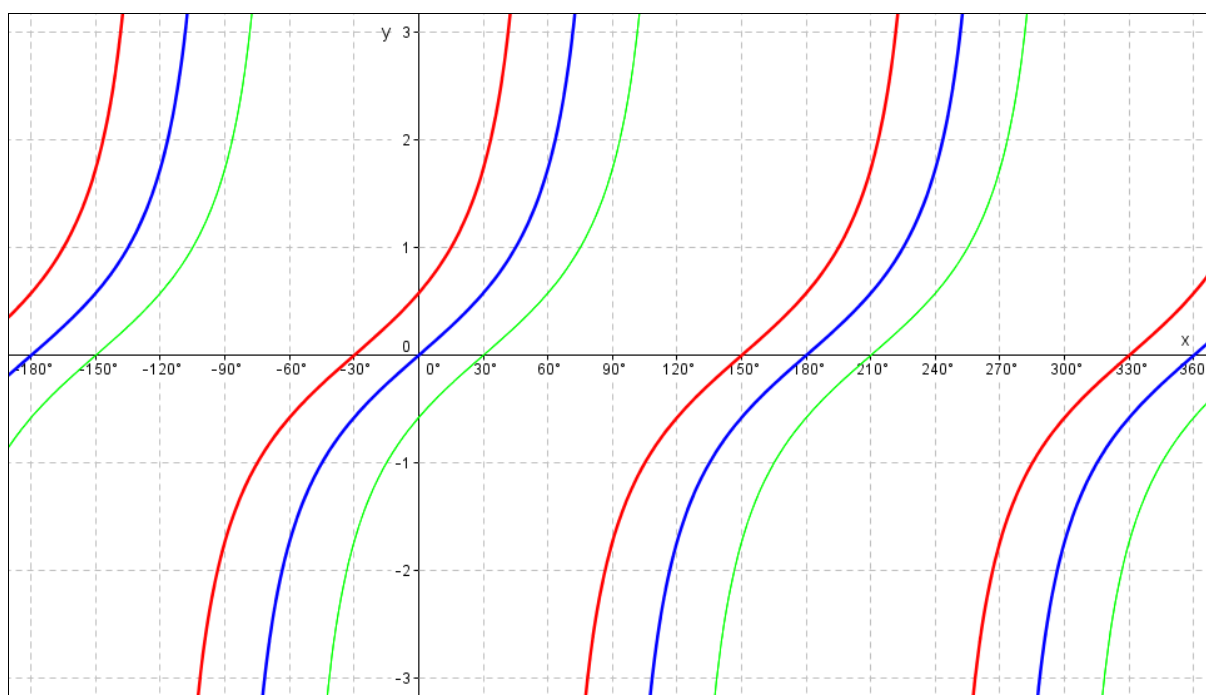
Graphs of the type $\{(x;y) : y = \tan(x + p)\}$

Sketch the following graphs on the same system of axes for $[0^\circ; 360^\circ]$.

$$y = \tan x$$

$$y = \tan(x + 30^\circ)$$

$$y = \tan(x - 30^\circ)$$



Completed table:

Function	Amplitude	Period	Zeros	Range
$y = \tan x$	-	180°	$0^\circ; 180^\circ; 360^\circ$	$[-\infty; \infty]$
$y = \tan(x + 30^\circ)$	-	180°	$-30^\circ; 150^\circ; 330^\circ$	$[-\infty; \infty]$
$y = \tan(x - 30^\circ)$	-	180°	$30^\circ; 210^\circ$	$[-\infty; \infty]$

Note :

$y = \tan(x - 30^\circ) \rightarrow$ graph of $y = \tan x$ translated 30° to the right (on the x-axis)

$y = \tan(x + 30^\circ) \rightarrow$ graph of $y = \tan x$ translated 30° to the left (on the x-axis)



Activity 3-5: Interpretation of graphs

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	60 min	- Whiteboard markers - Flipchart paper	None
In your groups you will: 1. Select a new scribe and a spokesperson for these activities. 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

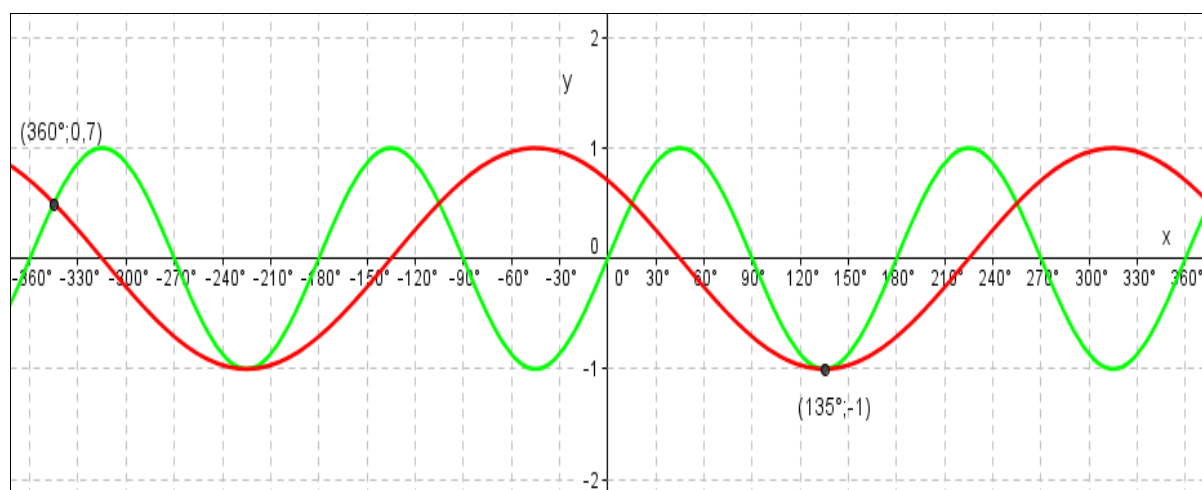
Activity 3

Make a study of all the trigonometric graphs in this module. Now complete the following table for $[-360^\circ; 360^\circ]$

Function	Amplitude	y – intercept	zeros (x – intercept)	period	asymptote / s	range
$y = \sin x + 1$						
$y = 3 \cos x$						
$y = 2 \tan x$						
$y = \sin(x + 20^\circ)$						
$y = \cos(x - 45^\circ)$						

Activity 4

The diagram below shows the graphs of $f(x) = \sin ax$ and $g(x) = \cos(x+b^\circ)$

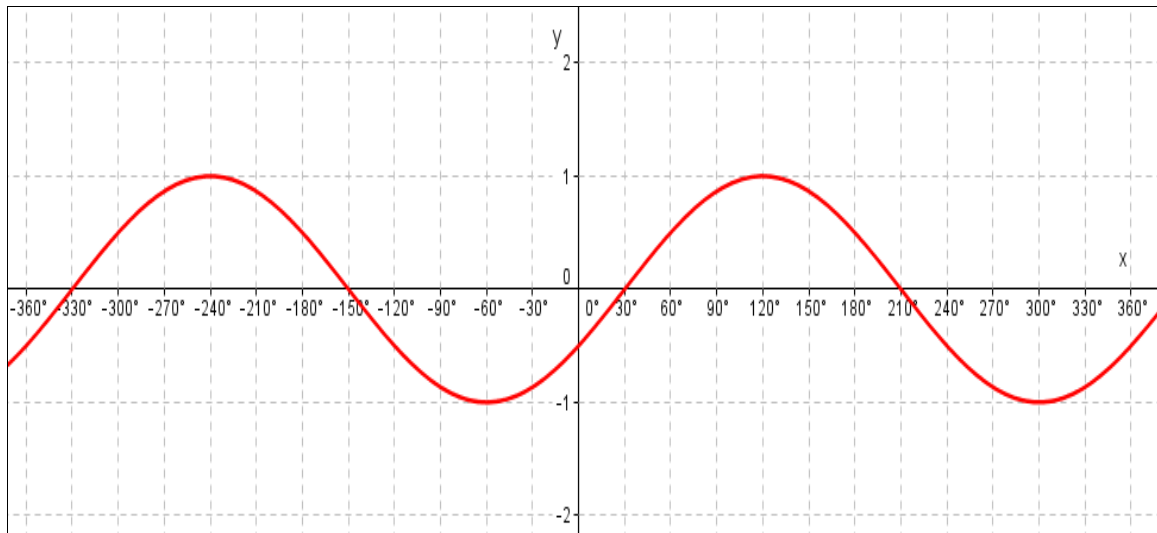


- 4.1 What is the period of f ?
- 4.2 Determine the values of a and b .
- 4.3 State the range of h if $h(x)=g(x)+1$.

Activity 5

Given the function:

$$f(x) = \sin(x - 30^\circ) \text{ for } x \in [-360^\circ; 360^\circ].$$



Determine:

- 5.1. The period of the function g , if $g(x) = f(2x)$.
- 5.2. The range of the function h , if $h(x) = f(x) - 1$.
- 5.3. The amplitude of the function q , if $q(x) = \frac{1}{2}f(x) + 2$.

Misconceptions

Learners should be aware that:

$$\sin 2x \neq 2 \sin x$$

$$\cos 2x \neq 2 \cos x$$

$$\tan 2x \neq 2 \tan x$$

Learners can be shown these differences.

For example choose $x = 60^\circ$,

$$\sin 2(60^\circ) = \sin 120^\circ = \frac{\sqrt{3}}{2}$$

$$2 \sin 60^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, $\sin 2x \neq 2 \sin x$

When working with compound angles it can be shown that:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \text{ (enrichment)}$$

Conclusion

Please consolidate the work done in this module by emphasising the following:

- How to calculate the periods of trigonometric functions.
- The tan graphs have no amplitude; the graphs have asymptotes (which are vertical).
- The various features should be identified and assist in the interpretation of such graphs.
- How graphs are translated along (a) the x-axis (b) the y-axis.

Module 2: Probability

Sub-topic 1: What is probability?

Grade 10	Grade 11	Grade 12
Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome. Venn diagrams as an aid to solving probability problems. Mutually exclusive events and complementary events. The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	Dependent and independent events. Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	Generalisation of the fundamental counting principle. Probability problems using the fundamental counting principle

Introduction

Probability is defined as a likelihood that a particular event would occur.

Before we go any further, why do we learn probability?

Probability is useful to humankind in that it helps make decisions by providing information on the likelihood of events.

What every day events use Probability? Name a few.

The weather report; insurance; lotto; medical research; law enforcement; and political science

Terminology

Trial: a systematic opportunity for an event to occur.

Experiment: one or more trials.

Sample space: the set of all possible outcomes of an event.

Event: an individual outcome or any specified combination of outcomes.

Complement: all outcomes that are not in the event but are in the universal set.

Complementary events: when two mutually exclusive events together contain all the outcomes in the sample space. For an event called A, we write the complement as “not A”. Another way of writing the complement is as A’.

Dependent events: if one event affects the occurrence of the other, then the events are dependent.

Independent events: two events are independent if the occurrence of one event has no effect on the likelihood of the occurrence of the other.

Frequency: The number of times a particular outcome occurs in a particular period or given time or is measured against a number of trials is the frequency of that outcome.

Incompatible events or mutually exclusive events: Two or more events are said to be mutually exclusive or incompatible when only one of those events can occur at a time.

Basic probability calculations

Probability is the study of how likely it is that a particular event will happen.

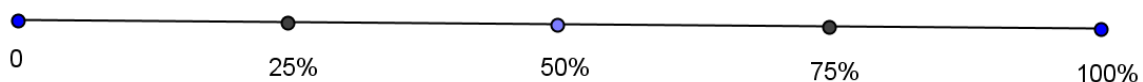
For example, these are questions about the probability of something happening.

- What is the chance that it will rain tomorrow?
- If you buy a Lotto ticket, what is the chance that you will win the Lotto?

Probability scale

The scale of chance, or likelihood, starts with 'no chance at all' and passes through various phases before getting to 'certain'. Some of those in-between phases could be 'almost impossible', 'unlikely', 'even chance' or 'highly likely'.

Scale	Never	Unlikely	Equal Chance	Likely	Always
	No Chance	Probably not	Even chance	Probably	Certain
	Impossible	Small chances of happening	Fifty-fifty	More chances of an event happening	Definite
Events	Getting a 12 from a pack of playing cards	Winning lotto	Heads when tossing a coin	Rain will fall in December in Free state	The sun will rise tomorrow



Relative frequency is approximated by performing and recording the ratio of the number of occurrences of the event to the number of trials. As the number of trials increases, the approximation of the experimental probability improves.

Consider this example:

What is the probability that when you look out of a window at a street, the next car that passes will be red? You can no longer use equally likely outcomes for this. You need to use relative frequency.

Theoretic probability is based on the assumption that all outcomes in the sample space occur randomly.

Consider this example:

When you toss a coin, there is an equal chance of obtaining heads or a tails. When you throw a die, the probability of getting a 6 is 1/6.

In your own words, what is the difference between relative frequency and theoretic probability?

For relative frequency in an event A:

$$\text{Probability that event } A \text{ occurs} = \frac{\text{number of occurrences of event } A}{\text{the number of trials}}$$

In theoretic probability the probability of an event A, denoted by P(A) is defined by:

$$P(A) = \frac{\text{number of outcomes in an event } A}{\text{number of outcomes in the sample space.}}$$



Worked Examples 1-3: Basic probability questions (25 Mins)

The facilitator will now guide and support you through these worked examples. Don't forget to raise your concerns and ask questions!

Example 1: Grade 10(R)

A bag contains 3 white cards, 2 black cards and 5 red cards. Find the probability of each event for one draw.

- 1.1 a white card
- 1.2 a black card
- 1.3 a red or black card

Solution:

$$\text{Probability of an event} = \frac{\text{number of required outcome}}{\text{number of possible outcome}}$$

$$1.1. P(\text{a white card}) = \frac{3}{10}$$

$$1.2. P(\text{a black card}) = \frac{2}{10} = \frac{1}{5}$$

$$1.3. P(\text{a red or a black}) = \frac{5+2}{10} = \frac{7}{10}$$

Example 2: Grade 11(p)

Create a tree diagram that shows the sample space for each event

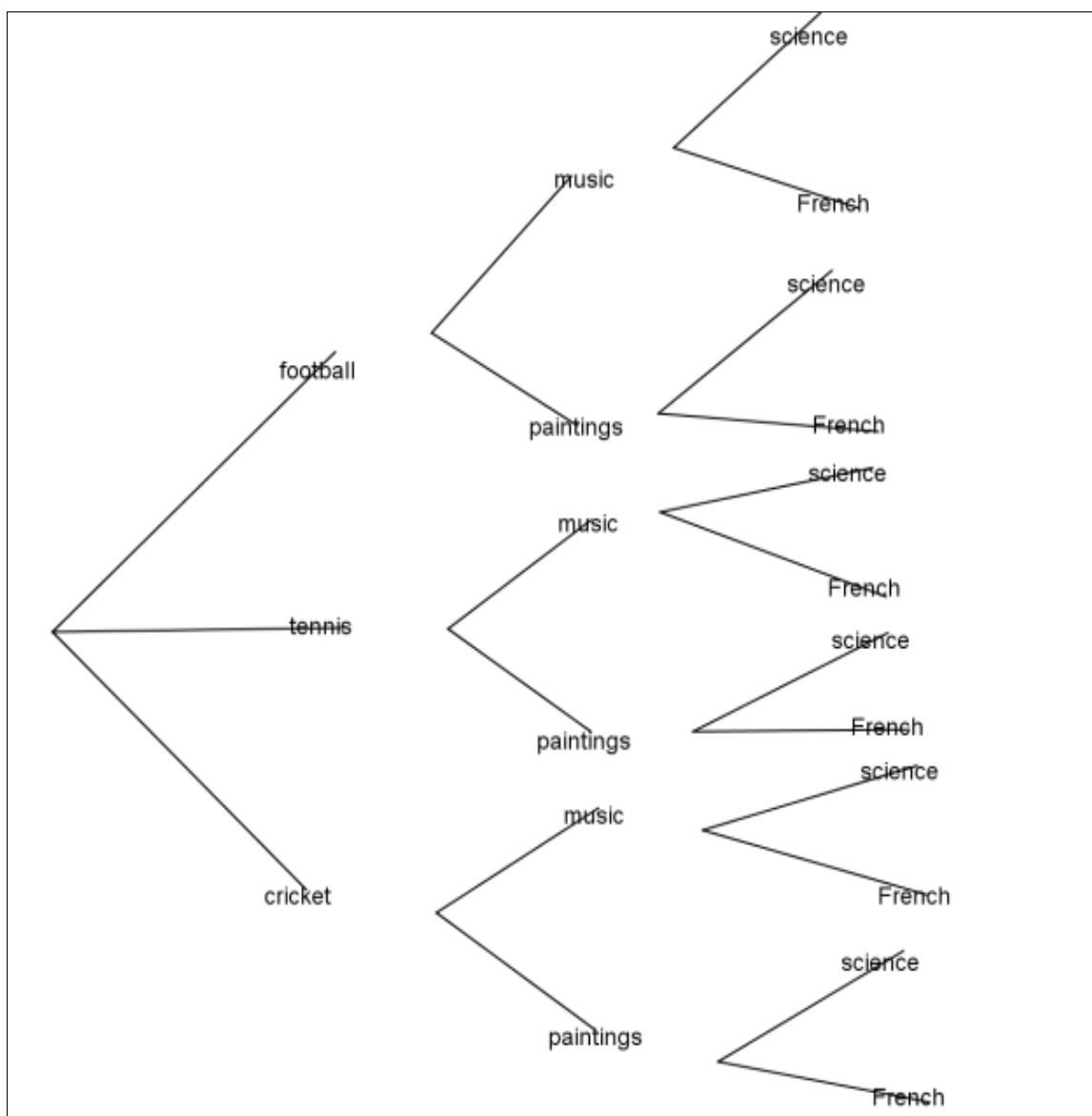
2.1 Involvement in one of each type of extracurricular activity

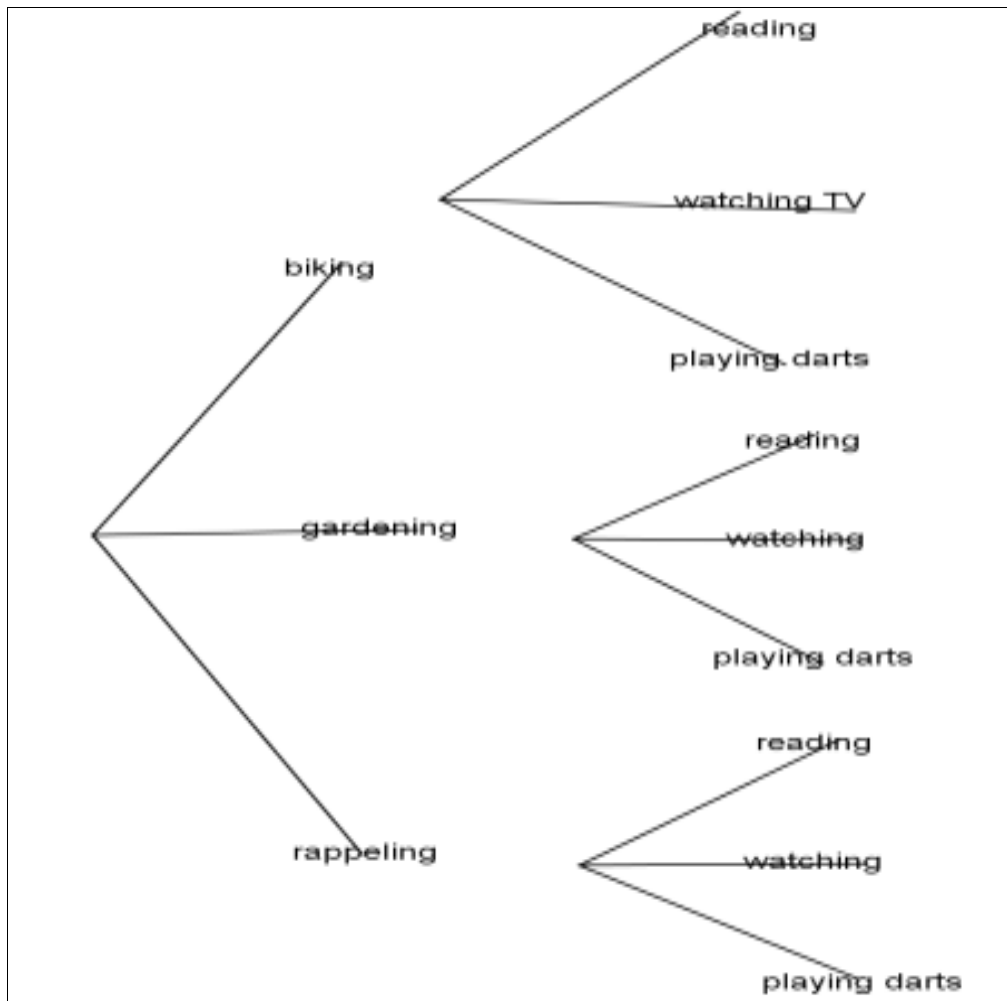
- Sports: football, tennis, cricket
- Arts: music, painting
- Clubs: science, French

2.2 Involvement in one of each type of leisure activity.

- Outdoor: biking, gardening, rappelling
- Indoor: reading, watching TV, Darts

Solution:





Example 3: Grade 12(C)

In a survey about a change in public policy, 100 people were asked if they favour the change, oppose the change, or have no opinion about the change. The responses are indicated below:

	Men	Women	Total
Favour	18	9	27
Oppose	12	25	37
No opinion	20	16	36
Total	50	50	100

3.1 Find the probability that a randomly selected respondent to the survey opposes or has no opinion about the change in policy.

3.2 Find the probability that a randomly selected respondent to the survey is a man or opposes the change in policy.

Solution:

3.1. The events “oppose” and “no opinion” are mutually exclusive events

$$\begin{aligned}
 P(\text{oppose or no opinion}) &= P(\text{oppose}) + P(\text{no opinion}) \\
 &= \frac{37}{100} + \frac{36}{100} \\
 &= \frac{73}{100} = 0,73 = 73\%
 \end{aligned}$$

3.2. The events “man and “oppose” are inclusive events

$$\begin{aligned}
 P(\text{man or oppose}) &= P(\text{man}) + P(\text{oppose}) - P(\text{man and oppose}) \\
 &= \frac{50}{100} + \frac{37}{100} - \frac{12}{100} \\
 &= \frac{75}{100} = \frac{3}{4} = 0,75 = 75\%
 \end{aligned}$$



Activity 1&2: Mutually exclusive and complementary events

Group organisation:	Time:	Resources:	Appendix:
Individual	15 min	• Participants Handout	None
Write the answers to the questions in the space provided.			

Activity 1: Grade 10

1.1. Explain the meaning of mutually exclusive events and of inclusive events.

1.2. Give 3 examples of each

Activity 2: Grade 10

2.1 Describe how to find the complement of the event 'rolling 1" or "rolling 2" on a die.

Conclusion

Probability is a section with few marks in our curriculum but these marks can make a difference between distinctions and passing. The focus should be on learners understanding terms and noticing the difference. Use real examples to teach this topic as it is applicable to real life.

Sub-topic 2: Venn Diagrams

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome. Venn diagrams as an aid to solving probability problems. Mutually exclusive events and complementary events. The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	Dependent and independent events. Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	Generalisation of the fundamental counting principle. Probability problems using the fundamental counting principle

Introduction

A Venn diagram is a diagram that shows all possible logical relations between finite collections of sets. Venn diagrams were conceived around 1880 by John Venn. Venn diagrams were a way of picturing relationships between different groups of things.

They are used to teach elementary set theory, as well as illustrate simple set relationships in probability. A universal set in a Venn diagram represents all the elements in consideration e.g. learners in a class; cars in the school car park; flowers in the flowerbed, etc. Elements in the universal set are then further classified into smaller sets or groups.



Worked Examples 1-3: Venn (30 Mins)

With your assistance, the facilitator will now demonstrate the use of some of these Venn diagrams.

Example 1: Grade 10(C)

In a bus there are 25 learners. 15 of them play hockey and 18 of them play basketball.

1.1 How many learners play both sports?

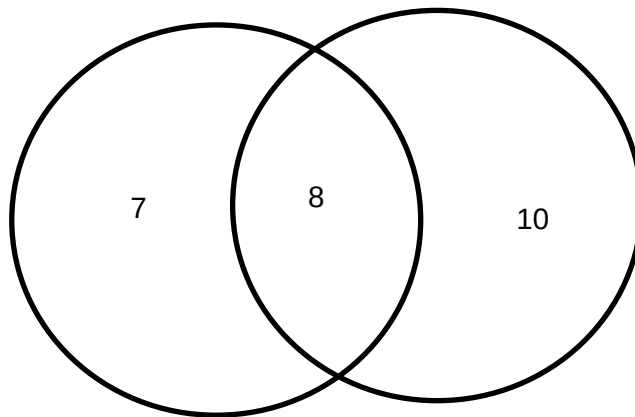
1.2 Illustrate the above information in a Venn diagram

1.3 Find the probability that a learner chosen at random plays basketball only

Solution:

1.1 number of learners that play both = $15 + 18 - 25$
 $= 8$

1.2 Illustration



1.3 Probability of basketball only = $\frac{\text{number of required outcome}}{\text{number of possible outcome}}$
 $= \frac{7}{25}$

Example 2: Grade 11(P)

The drama, mathematics and jazz clubs at Aden High School have 32 members, 33 members and 39 members respectively. 8 members belong to the three clubs; 19 members belong to both jazz and drama and 18 belong to mathematics and jazz, 6 belong to drama only and members exclusive to mathematics are 8. Illustrate this information in a Venn diagram.

Solution:

$n(D, J \text{ and } M \text{ intersect}) = 8$

$n(D \text{ and } J \text{ intersect}) = 19$ meaning that 11 members are exclusive to drama and jazz.

$n(J \text{ and } M \text{ intersect}) = 18$ implying that 10 members are exclusive to jazz and mathematics.

$n(J) = 39$ then let the number members exclusive to Jazz be x .

$$39 = x + 11 + 8 + 10$$

$$39 = x + 29$$

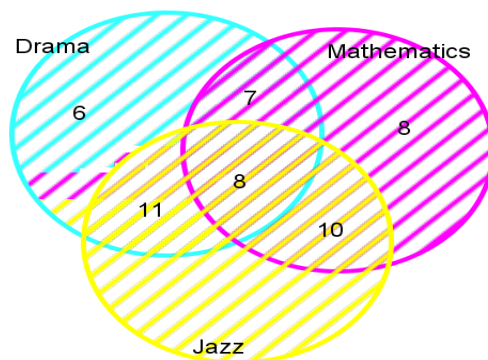
$\therefore x = 10$.

Let the number of members exclusive to drama and mathematics be y

$$y + 11 + 6 + 8 = 32$$

$$y + 25 = 32$$

$$\therefore y = 7$$



Example 3: Grade 12(P)

The science, art and computer clubs have 45 members, 38 members and 54 members respectively. If members that belong to all three clubs is x , members in both science and computer are 22, members in both computer and art are 21, 16 members are in science only and 20 members in computer club only. Solve for x and then complete the Venn diagram.

Solution:

$$22 - x + x + 21 - x + 20 = 54$$

$$63 - x = 54$$

$$x = 9$$

Members exclusive to the intersection of science and computer = $22 - 9 = 13$

Member's exclusive to the intersection of computers and arts is $21 - 9 = 12$

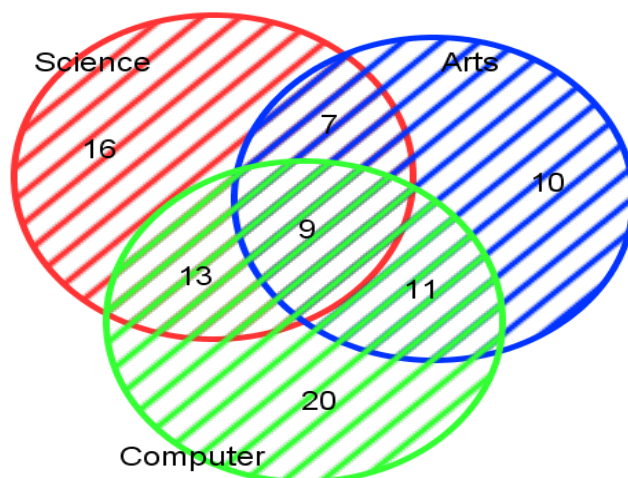
Let the number of members in science intersecting arts be m .

Then $m = 45 - (16 + 13 + 9)$ thus $m = 7$

Let the number of members exclusive to arts be n then

$$n = 38 - (7 + 9 + 12)$$

$$n = 10$$





Activities 1-3: Venn Diagrams

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	Participants Handout	None
In your groups you will: 1. Select a new scribe and a spokesperson for these activities. 2.2.1.1 Use the flipchart and permanent markers and answer the questions as per the activity.			

Activity 1: Grade 10

The band at Morningside High School has 50 members, and the student council has 20 members. If x members are common to both groups and those in student council only are 15.

- 1.1 Find the value of x
- 1.2 Illustrate this information in a Venn diagram
- 1.3 Find the probability that a learner chosen at random is not in the student council

Activity 2: Grade 11 & 12

- 2.1 How many integers from 1 to 600 are divisible by 2 or by 3? Now find the probability that a random integer from 1 to 600 is divisible by neither 2 nor 3.
- 2.2 How many integers from 1 to 3500 are divisible by 5 or 7? Now find the probability that a random integer from 1 to 3500 is divisible by neither 5 nor 7

Conclusion

Venn diagrams are easy to use and they make information easy to work with. First establish if the events are mutually exclusive or not and then determine the number of elements in the intersection of the two sets. Make sure you don't count elements twice, subtracting the number of elements that are in the intersection from the number of elements in that set to get the number of elements that are exclusive to that set.

Sub-topic 3: Dependent and independent events

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome. Venn diagrams as an aid to solving probability problems. Mutually exclusive events and complementary events. The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	Dependent and independent events. Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	Generalisation of the fundamental counting principle. Probability problems using the fundamental counting principle.

Introduction

If one event does affect the occurrence of the other event, the events are **dependent**. Two events are **independent** if the occurrence or non-occurrence of one event has no effect on the likelihood of the occurrence of the other event. For example, tossing two coins is an example of a pair of independent events.

Probability of Independent Events

Events A and B are independent events if and only if $P(A \text{ and } B) = P(A) \times P(B)$. Otherwise, A and B are dependent events.



Worked Examples 1-2: Dependent and independent events (30 Mins)

Allow the facilitator some time to guide and support you through the following worked examples. Don't forget to correct him/her if they err.

Example 1: Grade 11(C)

Suppose that the probability of Kevin coming to a party is 80% and the probability of Judy coming to a party is 95%. Assuming that these events are independent, what is the probability that:

- 1.1 They both come to the party.
- 1.2 Kevin comes but Judy does not come to the party.
- 1.3 They both don't come to the party.

Solution:

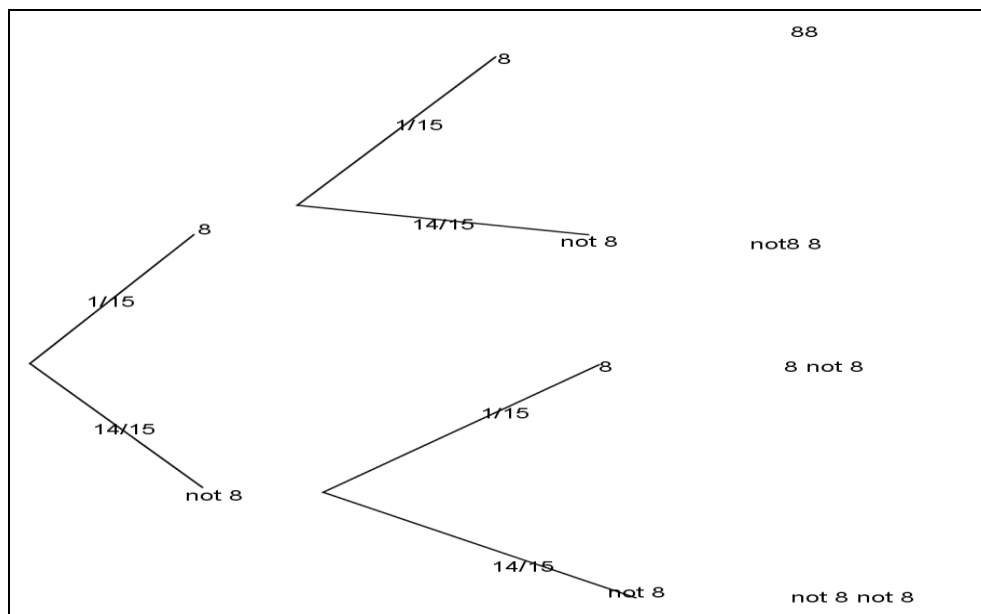
- 1.1 $P(\text{they both come})$
 $= 80\% \times 95\%$
 $= 76\% = 0,76$
- 1.2 $P(\text{Kevin comes, Judy does not})$
 $= 80\% \times 5\%$
 $= 4\% = 0,04$
- 1.3 $P(\text{They both don't come})$
 $= 20\% \times 5\% = 1\% = 0.01$

Example 2: Grade 12(C)

The integers 1 through 15 are written on slips of paper and placed into a box. One slip is selected at random and put back into the box, and then another slip is chosen at random.

- 2.1 What is the probability that the number 8 is selected both times?
- 2.2 What is the probability that the number 8 is selected exactly once?

Solution:



- $P(8 \text{ and } 8) = \frac{1}{15} \times \frac{1}{15} = \frac{1}{225}$
- $P(8 \text{ and not } 8 \text{ or not } 8 \text{ and } 8) = \frac{1}{15} \times \frac{14}{15} + \frac{14}{15} \times \frac{1}{15} = \frac{28}{225}$



Activity 1: Simple Calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	15 min	Participants Handout	None

In your groups you will:

1. Select a new scribe and a spokesperson for these activities.
2. Use the flipchart and permanent markers and answer the questions as per the activity.

Activity 1

Suppose that a security system consists of four components: a motion detector, a glass-break detector, magnetic door and window contacts and a video camera. The probability of escaping detection by each of the four devices are 0,2; 0,3; 0,4 and 0,6 respectively. Assuming that all of the components act independently:

- 1.1 What is the probability that a thief can get past the magnetic contacts and the video camera?
- 1.2 What is the probability that a thief can get past the glass-break detector and the motion detector?
- 1.3 What is the probability that a thief can get past all four components?
- 1.4 Comment on the chances of burglary.

Conclusion

Dependent events are probabilities that change when one event occurs, for example, in a pack of 52 cards, the probability of getting a diamond given that the first card drawn is an ace. The probability of getting a diamond is now based on 51 cards. Independent events are not affected by the previous events, for example, getting a head from a toss of a coin.

Sub-topic 4: The fundamental counting principle

CAPS extraction indicating progression from grades 10-12

Grade 10	Grade 11	Grade 12
Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome. Venn diagrams as an aid to solving probability problems. Mutually exclusive events and complementary events. The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	Dependent and independent events. Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	Generalisation of the fundamental counting principle. Probability problems using the fundamental counting principle.

Introduction

A **permutation** is an arrangement of objects in a specific order. When objects are arranged in a row, the permutation is called a **linear permutation**. Unless otherwise noted, the term permutation will be used to mean linear permutations

Permutations of n Objects

The number of permutations of n objects is given by n!

Permutations of n objects taken r at a time

The number of permutations of n objects taken at a time, denoted by P(n, r) or nPr is given by $P(n, r) = nPr = \frac{n!}{(n-r)!}$, where $r \leq n$

Permutations with identical objects

The number of distinct permutations of n objects with r identical objects is given by $\frac{n!}{r!}$ where $1 \leq r \leq n$.

The number of distinct permutations of n objects with r_1 identical objects, r_2 identical objects of another kind, r_3 identical objects of another kind, . . . , and r_k identical objects of another kind is given by

$$\frac{n!}{r_1! r_2! r_3! \dots r_k!}$$

If n is a positive integer, and then n factorial, written $n!$, is defined as follows:

$$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$$

Note that the value of $0!$ is defined to be 1

Teaching tip: use pencils of different colours to show practical arrangement of items and the proof of the factorial notation.



Worked Examples 1-2: Counting Principles(30 Mins)

Allow the facilitator some time to guide and support you through the following worked examples. Don't forget to correct him/her if they err.

Example 1:

In how many ways can the letters ABCD be arranged?

Solution:

$$\begin{aligned} n! &= 4! \\ &= 4 \times 3 \times 2 \times 1 \\ &= 24 \end{aligned}$$

Example 2:

In 12-tone music, each of the 12 notes in an octave must be used exactly once before any are repeated. A set of 12 tones is called a tone row. How many different tone rows are possible?

Solution:

Find the number of permutations of 12 notes

$$\begin{aligned} 12! &= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 479\,001\,600. \end{aligned}$$

There are 479 001 600 different tone rows for 12 tones.

Example 3:

Find the number of ways to listen to 5 different CDs from a selection of 15 CDs.

Solution:

Find the number of permutations of 15 objects taken 5 at a time.

$${}_{15}P_5 = \frac{15!}{(15-5)!} = \frac{15 \times 14 \times 13 \times 12 \times 11}{10!}$$

$$= 15 \times 14 \times 13 \times 12 \times 11$$

$$= 360\,360$$

Example 4:

Find the probability of drawing 5 cards, all hearts, from a standard deck of cards.

Solution:

The probability of drawing a heart on the first draw is $\frac{13}{52}$, then next will be $\frac{12}{51}$ and so on until the fifth card.

$$P(5 \text{ hearts in a row}) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}$$

$$= \frac{33}{66\,400}$$

 Activity 1 & 2: Counting Principles			
Group organisation:	Time:	Resources:	Appendix:
Pairs	20 min	Participants Handout	None
In pairs answer the questions that follow. Both participants are expected to write their answers in the space provided.			

Activity 1:

In how many ways can the letters in the word PENCIL be arranged?

Activity 2:

What is the probability that a random arrangement of the letters BAFANA starts and ends with an 'A'?

Conclusion

The counting principle is very important as it helps learners as they move into higher education. This is not a difficult topic and learners can capitalise by scoring the maximum marks. The most important thing is to make learners understand key words and the correct formula.

Module 3: Statistics

(Due to the nature of this topic, it will be handled per grade and not per sub-topic. Please note that much of the examples can be integrated from grade to grade. This was done as the difficult topics per grade were identified and addressed.)

Grade 10	Grade 11	Grade 12
Collect, organise and interpret univariate numerical data in order to determine: • measures of central tendency (mean, median, mode) of grouped and ungrouped data and represent these by five-number summary (maximum, minimum, quartiles) and box and whisker diagrams, and know which is the most appropriate under given conditions; • measures of dispersion: percentiles, quartiles, deciles, interquartile and semi-inter-quartile range. Identify possible sources of bias and errors in measurements.	Represent data effectively, choosing appropriately from: • bar and compound bar graphs; • histograms (grouped data); • frequency polygons; • pie charts; • line and broken line graphs. Represent measures of central tendency and dispersion in univariate numerical data by: • using ogives; • calculating the variance and standard deviation of sets of data manually (for small sets of data) and using available technology (for larger sets of data) and representing results graphically. Skewed data in box and whisker diagrams and frequency polygons. Identify outliers.	Represent bivariate numerical data as a scatter plot and suggest intuitively and by simple investigation whether a linear, quadratic or exponential function would best fit the data. Use of available technology to calculate the linear regression line which best fits a given set of bivariate numerical data. Use of available technology to calculate the correlation co-efficient of a set of bivariate numerical data and make relevant deductions.

Introduction

Information in the form of numbers, graphs and tables is all around us; on television, on the radio or in the newspaper. We are exposed to crime rates, sports results, rainfall, government spending, and the rate of HIV/AIDS infection, population growth and economic growth.

This hand-out demonstrates how Mathematics can be used to manipulate data, to represent or misrepresent trends and patterns, and to provide solutions that are directly applicable to the world around us. Skills relating to the collection, organisation, display, analysis and interpretation of information that were introduced in earlier grades are developed further.

Prior knowledge

- **Collecting data:** including distinguishing between samples and populations
- **Organising and summarising data:** using tallies, tables and stem-and-leaf displays; determining measures of central tendency (mean, median, mode); determining measures of dispersion (range, extremes, outliers)
- **Representing data:** drawing and interpreting bar graphs, double bar graphs, histograms, pie charts, broken-line graphs, scatter plots.
- **Interpreting data:** critically reading and interpreting two sets of data represented in a variety of graphs.
- **Analysing data:** critically analysing data by answering questions related to data collection methods, summaries of data, sources of error and bias in the data
- **Reporting data:** by drawing conclusions about the data; making predictions based on the data; making comparisons between two sets of data; identifying sources of error and bias in the data; choosing appropriate summary statistics (mean, median, mode, range) for the data and discussing the role of extremes and outliers in the data.

New terminology

- A **measure of central tendency** is an estimation of a middle or central value of a data set.
- There are three measures of central tendency, namely the mean, median and mode:
- The **mean** = sum of all the values $\div$ total number of values.
- The **median** is the middle value of a data set after ordering the data. If there is an even number of values, there are two middle values, so the median will lie in between those two values.
- The **mode** is the value that occurs most often.
- The **lower quartile** is the middle of the lower half and is the $(n + 1) / 4$ values. It is the same as the 25th percentile
- The **upper quartile** is the middle of the upper half and is the $3(n + 1) / 4$ values. It is the same as the 75th percentile.
- The **interquartile range** is upper quartile-lower quartile. This is the same as the range of the middle 50%.

Misconceptions

Common errors and misconceptions

- a) Learners do not realise that the minimum information required for drawing a box and whisker diagram is a five-number summary. Many learners were unable to interpret the box and whisker diagram. They did not realise that quartiles divide a data set into four quarters, each of which has 25% of the data. There is confusion between quartiles and quarters.
- b) Learners do not understand the concept of inter-quartile range. Some learners confused inter-quartile range with semi-inter-quartile range and others with range.
- c) Learners' misconceptions are grounded in poor language background. Learners had difficulty in decoding 'in excess of' and 'least variation'.
- d) There is a misconception that the line of best fit is a straight line. Many learners draw a straight line instead of an exponential curve as a line of best fit. Some learners connect all the points by using a ruler.
- e) Learners did not know the meaning of the word 'trend'. Some confuse it with 'correlation'. Learners have difficulty in describing the trend between the two variables with many of them providing incomplete answers or not explicitly describing the trend as exponential.
- f) Learners are unable to read correctly off the ogive. Learners are unable to write the interval in correct notation. While learners might be familiar with the concept of mode or modal class if the data is given in table form, learners have difficulty in identifying the modal class from the ogive.
- g) Learners are unable to see the link between the information provided in the table and the distribution curves.

Suggestions for improvement

- a) Statistical literacy is not just about procedures to calculate an answer or to draw a graphical representation. Interpreting data and drawing appropriate and valid conclusions is much needed in everyday life. Hence, analysing data must be the focus of teaching and learning. Learners need to be made aware that a box and whisker diagram is drawn using the values in the five-number summary. In preparation for this, teachers should not only ask learners to identify the quartiles in a data set but also to write down the five-number summary for the data set. The emphasis in teaching the box and whisker diagram is on its representing the data set as four sections, each of which contains 25% of the data. But more than that, the space between the values tells us how closely together or widely spread the data are for that section. Once learners have understood the idea of the spread of data in each section of the box and whisker diagram, the teacher should then focus on the idea of skewedness. Learners should also be shown how to draw a box and whisker diagram from an ogive.
- b) Teachers need to emphasise the terminology used in Statistics and explain the meaning of terms used.
- c) If learners are studying in a language that is not their home language, teachers need to ensure that the concepts used in Mathematics are thoroughly explained to them. Teachers need to explain the different terms that learners will encounter in this topic. Poor understanding of these terms is a contributory factor to the learners' confusion.
- d) The concept: the line of best fit should not be taken literally to mean a straight line only. The line of best fit includes a straight line, a quadratic or an exponential curve. In the

case of the quadratic or exponential curve, the line of best fit is a smooth curve and learners are advised not to use a ruler to draw these curves. When teaching scatter plots, teachers should include examples where the line of best fit is quadratic or exponential and not only focus on those where the line of best fit is a straight line.

- e) Learners must be encouraged to be explicit in describing the trend between variables. Merely stating that there is an increasing trend could imply a linear relationship between the variables. However, stating that as one variable increases the other increases rapidly distinguishes an exponential situation from the linear case. Teachers are encouraged to experiment with data sets. Allow learners to interpret the data sets for trends and other salient features. These interpretations should be debated vigorously in the class to ensure that the conclusions reached are valid. Learners should be encouraged to think critically and not be afraid of making mistakes.
- f) As a starting point, learners need to realise that the cumulative frequency is the total of all the frequencies up to that point. The ogive is a graphical representation of a cumulative frequency table. Learners should exercise care when reading off a graph. If need be, they should use a ruler to ensure more accurate readings. While it is a routine task to draw an ogive from a cumulative frequency table, it is advisable that teachers also demonstrate how the cumulative frequency table can be derived from the ogive and, to take it slightly further, how the frequency table can be constructed from a cumulative frequency table. The frequency table should be viewed alongside the ogive. In this way, learners should be able to identify the modal class in the ogive more readily as the part of the curve where the gradient is the steepest.

Grade 10



Worked Examples (30 Min)

The facilitator will discuss the examples with you and provide you with possible ways in which to teach your learners some of these topics. If you know of a better way in which this can be taught, please do not hesitate to make your expertise known. We can all learn from each other.

Example 1: Measures of dispersion

What are the quartiles of {3; 5; 1; 8; 9; 12; 25; 28; 24; 30; 41; 50}?

Solution:

Step 1: Order the data set from lowest to highest.

{1; 3; 5; 8; 9; 12; 24; 25; 28; 30; 41; 50}

Step 2: Count the number of data values in the data set.

There are 12 values in the data set.

Step 3: Divide the number of data values by 4 to find the number of data values per quartile.

$$12 \div 4 = 3$$

Step 4: Find the data values corresponding to the quartiles.

1 3 5 (Q1) 8 9 12 (Q2) 24 25 28 (Q3) 30 41 50

The first quartile occurs between data position 3 and 4 and is the average of data values 5 and 8. The second quartile occurs between positions 6 and 7 and is the average of data values 12 and 24. The third quartile occurs between positions 9 and 10 and is the average of data values 28 and 30.

Step 5: Answer

The first quartile = 6,5 (Q1)

The second quartile = 18 (Q2)

The third quartile = 29 (Q3)

Example 2: Measures of dispersion

A certain school provides buses to transport the learners to and from a nearby village.

A record is kept of the number of learners on each bus for 26 school days.

{27; 5; 27; 29; 31; 24; 25; 27; 28; 29; 24; 26; 30; 28; 31; 25; 25; 27; 28; 28; 28; 26; 28; 31; 24; 30}

2.1 Organise the data in a frequency table.

2.2 Use the table to calculate the total number of learners that were transported to school by bus.

2.3 Calculate the mean number of learners per trip, correct to one decimal place.

2.4 Explain what the mean represents.

2.5 Find the mode.

2.6 Find the median and explain what the median represents.

Solution:

2.1

Number of learners per trip (x)	Frequency	$f \times x$
24	3	$3 \times 24 = 72$
25	4	$4 \times 25 = 100$
26	2	$2 \times 26 = 52$
27	4	$4 \times 27 = 108$
28	6	$6 \times 28 = 168$
29	2	$2 \times 29 = 58$
30	2	$2 \times 30 = 60$
31	3	$3 \times 31 = 62$
	$n = 26$	$\sum f \cdot x = 711$

2.2 Total number of learners that were transported to school by bus = 711

2.3 Mean number of learners on each bus = $\frac{\text{total number of learners on all the bus trips}}{\text{total number of bus trips}}$

$$\begin{aligned}
&= \frac{\sum fx}{n} \\
&= \frac{711}{26} \\
&= 27,346 \\
&\approx 27,3
\end{aligned}$$

2.4 The mean tells us that if each bus had exactly the same number of people each time, there would be approximately 27 learners on each bus.

2.5 The largest value in the frequency column is 6 and it goes with 28 learners. This means that the mode = 28 learners.

2.6 There are 26 bus trips.

Half of 26 = 13, so the median lies between the 13th and the 14th values on the table.

Number of learners per trip	Frequency	Cumulative frequency
24	3	3
25	4	3 + 4 = 7
26	2	7 + 2 = 9
27	4	9 + 4 = 13
28	6	13 + 6 = 19
29	2	19 + 2 = 21
30	2	21 + 2 = 23
31	3	23 + 3 = 26

So the 13th value is 27 and the 14th value is 28.

$$\text{Median} = \frac{27+28}{2} = \frac{55}{2} = 27,5$$

The median tells us that for half of the bus trips there were less than 27,5 learners (which means 27 and less) on the bus, and for half of the bus trips there were more than 27,5 learners (which means 28 or more) on the bus.

Using a scientific calculator to find the mean of data:

- A scientific calculator makes it quicker and easier to find the mean of data in a frequency table.
- The key sequences for the CASIO fx-82ZA PLUS and the SHARP EL-W535HT that can be used to find the mean are as follows:

CASIO
- First add in a frequency column: [SHIFT] [SETUP] [3:STAT] [1:ON] ↓ - Then enter the data [SETUP] [2:STAT] [1:1-VAR] 24 [=] 25 [=] 26 [=] 27 [=] 28 [=] 29 [=] 30 [=] 31 [=] ↵ ⇨ 3 [=] 4 [=] 2 [=] 4 [=] 6 [=] 2 [=] 2 [=] 3 [=] [AC] [SHIFT] [STAT] [1] [4:VAR] [2 :$\bar{x}$]
SHARP
[MODE] [1 : STAT] [0 : SD] [2ndF] [MODE] [CA] 24 [(x ; y)] 3 [DATA] 25 [(x ; y)] 4 [DATA] 26 [(x ; y)] 2 [DATA] 27 [(x ; y)] 4 [DATA] 28 [(x ; y)] 6 [DATA] 29 [(x ; y)] 2 [DATA] 30 [(x ; y)] 2 [DATA] 31 [(x ; y)] 3 [DATA] [ALPHA] [4] [$\bar{x}$]

Activity 3: Measures of central tendency – frequency tables (grouped data)

The year-end examination marks (as a percentage) for 103 Grade 10 learners are recorded and grouped into intervals of 20 marks for analysis.

Study the data and answer the questions that follow.

Marks	Frequency (f)	Midpoint of class interval (x)	f·x
$0 \leq x < 20$	8		
$20 \leq x < 40$	12		
$40 \leq x < 60$	44		
$60 \leq x < 80$	27		
$80 \leq x < 100$	12		
Total:	$n = \sum f =$		$f \cdot x =$

- 3.1 Complete the given table.
 3.2 Estimate the mean for this set of marks.
 3.3 Give the modal class interval.
 3.4 Estimate the mode.
 3.5 Determine in which interval the median lies.
 3.6 In which interval do the following lie?
 (i) The lower quartile
 (ii) The upper quartile
 (iii) The 90th percentile

Solution:

3.1 Table

Marks	Frequency (f)	Midpoint of class interval (x)	f·x
$0 \leq x < 20$	8	10	80
$20 \leq x < 40$	12	30	360
$40 \leq x < 60$	44	50	2200
$60 \leq x < 80$	27	70	1890
$80 \leq x < 100$	12	90	1080
Total:	$n = \sum f = 103$		$f \cdot x = 5610$

$$\begin{aligned}
 3.2 \quad \bar{x} &= \frac{\sum f(x)}{\sum f} \\
 &= \frac{5610}{103} \\
 &= 54,5
 \end{aligned}$$

3.3 The modal class is the class with the largest frequency.
 The largest frequency is 44, and so the modal class is $40 \leq x < 60$.

3.4 We estimate the mode using the midpoint of the modal class. The mode is 50.

3.5 The position of the median is:

$$\begin{aligned}\frac{n+1}{2} \\&= \frac{103+1}{2} \\&= \frac{104}{2} \\&= 52\end{aligned}$$

3.6 i) The position of the lower quartile is:

$$\begin{aligned}\frac{1}{4}(n+1) \\&= \frac{1}{4}(103+1) \\&= 26\end{aligned}$$

Therefore, the lower quartile is in the 26th position, and lies in the interval $40 \leq x < 60$.

ii) The position of the upper quartile is:

$$\begin{aligned}\frac{3}{4}(n+1) \\&= \frac{3}{4}(103+1) \\&= 78\end{aligned}$$

Therefore, the upper quartile is in the 78th position, and lies in the interval $60 \leq x < 80$.

ii) The position of the 90th percentile is:

$$\begin{aligned}\frac{90}{100}(n+1) \\&= \frac{90}{100}(103+1) \\&= 93,6\end{aligned}$$

Therefore, if we round this up, we see that the 90th percentile lies in 94th position, which is in the interval $80 \leq x < 100$.



Activities 1&2: Some Calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	25 min	• Flipchart paper • Whiteboard markers	None
In your groups you will: 1. Select a new scribe and a spokesperson for these activities. 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

Activity 1: Measures of dispersion

12 Learners write a test. The results are as follows:

20, 39, 40, 43, 43, 46, 53, 58, 63, 70, 75, 91.

Find the range, quartiles and the interquartile range.

Activity 2: Measures of central tendency

There are four children in a family. The two oldest children are twins. The mean of the four children's ages is 14,25 years, the median is 15,5 years and the mode is 16 years. Use this information to work out the ages of the four children.

Grade 11



Worked Examples (30 Min)

The facilitator will discuss the examples with you and provide you with possible ways in which to teach your learners some of these topics. If you know of a better way in which this can be taught, please do not hesitate to make your expertise known. We can all learn from each other.

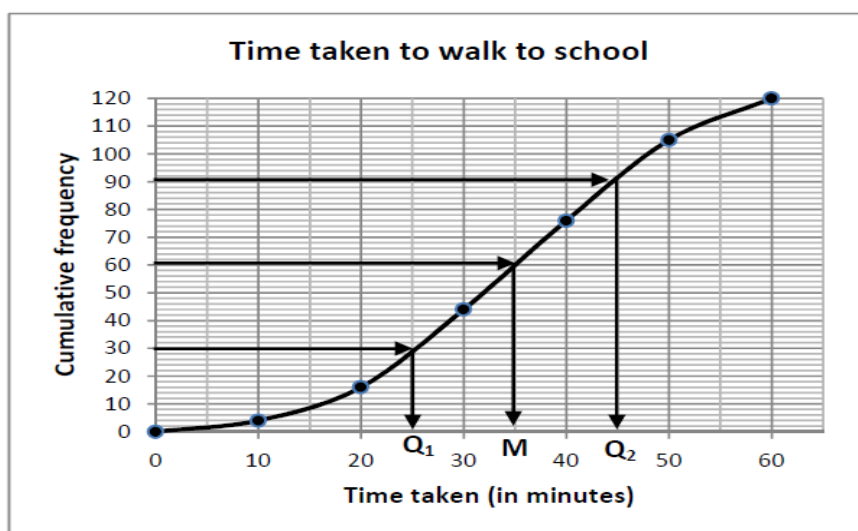
Example 1: Frequency tables – cumulative frequency- ogives

1. Use the ogive drawn to:

1.1 Determine the approximate values of:

- i) the median
- ii) the lower quartile
- iii) the upper quartile of the set of data.

1.2 What does each of these values tell you about the time taken by the learners?



Solution:

- 1.1 i) To find the approximate value of the **median (M)**, find the midpoint of the values plotted on the **cumulative frequency axis**. The maximum value is 120, so the median lies between the 60th and 61st term. Draw a horizontal line from just above 60 until it touches the ogive. From that point, draw a vertical line down to the **horizontal axis**. So the **median ≈ 35 min**
 - ii) To find the approximate value of the **lower quartile (Q1)**, find the midpoint of the lower half of the values plotted on the **cumulative frequency axis**. There are 60 terms in the lower half of the data, so the lower quartile lies between the 30th and the 31st term. Draw a horizontal line from just above 30 until it touches the ogive. From that point, draw a vertical line down to the **horizontal axis**. So the **lower quartile ≈ 25 min**.
 - iii) To find the approximate value of the **upper quartile (Q3)**, find the midpoint of the upper half of the values plotted on the **cumulative frequency axis**. There are 60 terms in the upper half of the data, so the upper quartile lies between $60 + 30 = 90$ th and the 91st term. Draw a horizontal line from just above 90 until it touches the ogive. From that point, draw a vertical line down to the **horizontal axis**. So the **upper quartile ≈ 45 min**.
- 1.2 i) The **median** tells us that 50% of the learners took 35 min or less or to walk to school.
 - ii) The **lower quartile** tells us that 25% of the learners took 25 min or less to walk to school.
 - iii) The **upper quartile** tells us that 75% of the learners took 45 min or less to walk to school.

Example 2: Variance and standard deviation of ungrouped data

- 2.1 Calculate the variance and the standard deviation of the following two data sets:
 Set A = {182; 182; 184; 184; 185; 185; 186}
 Set B = {152; 166; 176; 184; 194; 200; 216}
- 2.2 Use the two standard deviations to compare the distribution of data in the two sets.

Solution:

2.1 Set A

Data Item	Deviation from the mean	(Deviation) ²
182	$182 - 184 = -2$	$(-2)^2 = 4$
182	$182 - 184 = -2$	$(-2)^2 = 4$
184	$184 - 184 = 0$	$(0)^2 = 0$
184	$184 - 184 = 0$	$(0)^2 = 0$
185	$185 - 184 = 1$	$(1)^2 = 1$
185	$185 - 184 = 1$	$(1)^2 = 1$
186	$186 - 184 = 2$	$(2)^2 = 4$
		$\sum (\text{deviations})^2 = 14$

$$\text{Variance} = \frac{\sum (\text{deviations})^2}{\text{number of deviations}} = \frac{14}{7} = 2$$

$$\text{Standard deviation} = \sqrt{\text{variance}} = \sqrt{2} = 1,414$$

2.3 Set B

Data Item	Deviation from the mean	(Deviation) ²
152	$152 - 184 = -32$	$(-32)^2 = 1024$
166	$166 - 184 = -18$	$(-18)^2 = 324$
176	$176 - 184 = -8$	$(-8)^2 = 64$
184	$184 - 184 = 0$	$(0)^2 = 0$
194	$194 - 184 = 10$	$(10)^2 = 100$
200	$200 - 184 = 16$	$(16)^2 = 256$
216	$216 - 184 = 32$	$(32)^2 = 1024$
		$\sum (\text{deviations})^2 = 2792$

$$\text{Variance} = \frac{\sum (\text{deviations})^2}{\text{number of deviations}} = \frac{2792}{7} = 398,857 \dots$$

$$\text{Standard deviation} = \sqrt{\text{variance}} = \sqrt{\frac{2792}{7}} = 19,971$$



Activities 1&2: Some Calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	• Flipchart paper • Whiteboard markers	None
In your groups you will: 1. Select a new scribe and a spokesperson for these activities. 2. Use the flipchart and permanent markers and answer the questions as per the activity.			

Activity 1: Standard deviation

The time (in minutes) taken by a group of athletes from Lesiba High School to run a 3 km cross country race is: {18; 21; 16; 24; 28; 20; 22; 29; 19; 23}

Use your calculator to determine:

- 1.1 The mean time taken to complete the race
- 1.2 The standard deviation of the time taken to complete the race
- 1.3 What the standard deviation means.

Activity 2: Cumulative frequency - ogives

Fifty learners who travel by car to school were asked to record the number of km travelled to and from school in one week. The following table shows the results:

Number of kilometres	Number of learners	Cumulative frequency
$10 < x \leq 20$		2
$20 < x \leq 30$		9
$30 < x \leq 40$		13
$40 < x \leq 50$		26
$50 < x \leq 60$		42
$60 < x \leq 70$		50
	Total =	

- 2.1 Copy the table and then fill in the second column of the table.
- 2.2 Draw an ogive to illustrate the data.
- 2.3 Use your graph to estimate the median number of km travelled per week.

Grade 12



Worked Examples (30 Min)

The facilitator will discuss the examples with you and provide you with possible ways in which to teach your learners some of these topics. If you know of a better way in which this can be taught, please do not hesitate to make your expertise known. We can all learn from each other.

Example 1: Correlation coefficient

In the 2011 Household Survey, a representative sample of people was asked how many rooms they have in their homes. The table below shows the data taken from the Gauteng Province:

Number of rooms per household		1	2	3	4	5	6	7	8	9	10
Percentage of the households		20,6	13,9	10,9	17,9	13,1	9,8	6,1	3,8	2,6	1,3

Use your calculator to find the correlation coefficient r .

Using the formula:

In Words	Symbols
Find the sum of the x -values	$\sum x$
Find the sum of the y -values	$\sum y$
Multiply each x -value by its corresponding y -value and then find the sum	$\sum xy$
Square each x -value and find the sum of the squares	$\sum x^2$
Square each y -value and find the sum of the squares	$\sum y^2$
Use these five sums to calculate the correlation coefficient	$r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n \sum x^2 - (\sum x)^2} \sqrt{n \sum y^2 - (\sum y)^2}}$

Solution:

1.1 First, get the calculator in STAT mode:

[MODE] [2: STAT]

[2: A + BX]

Enter the x-values:

1 [=] 2 [=] 3 [=] 4 [=] 5 [=] 6 [=]

7 [=] 8 [=] 9 [=] 10 [=]

Enter the y-values

⇩ ⇨

20,6 [=] 13,8 [=] 10,9 [=] 17,9 [=]

13,1 [=] 9,8 [=] 6,1 [=] 3,8 [=] 2,6 [=]

1,3 [=]

[AC]

Find the value for r

[SHIFT] [1:STAT]

[5:Reg]

[3: r] [=]

$r = -0,917\ 952\ 1043$

Example 2: Regression line

The table below shows the masses and the heights of seven learners.

Mass	49	65	82	60	65	94	88
Height (cm)	15	17	18	15	16	19	180
	5			3	3	2	

2.1 Use your calculator to determine r , the correlation coefficient.

2.2 Use the method of least squares to determine the equation of the regression line.

2.3 Draw a scatter plot to illustrate the data.

2.4 Draw the regression line on the scatter plot.

2.5 Use your graph to determine Joan's height if her mass is 50 kg.

Solution:

2.1 First get the calculator in STAT mode:

[MODE] [2: STAT] [2: A + BX]

• Enter the x-values:

49 [=] 65 [=] 82 [=] 60 [=] 65 [=]

94 [=] 88 [=]

• Enter the y-values

⇩ ⇨

156 [=] 176 [=] 183 [=] 153 [=] 163

[=] 192 [=] 180 [=]

[AC]

• Find the value for **a**, the y-intercept

[SHIFT] [1:STAT]

[5:Reg] [1: A] [=]

a = 113,4716211 » 113,47.

• Get the value for **b**, the gradient

[SHIFT][1:STAT]

[5:Reg] [2: B] [=]

b = 0,812522171 » 0,81

2.2

First calculate $\bar{x}$ and $\bar{y}$, then find $\sum(x - \bar{x})(y - \bar{y})$ and $\sum(x - \bar{x})^2$.

It helps to draw up a table and to fill everything onto it:

mass	height				
x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$
49	156	-22,86	-15,86	362,56	522,58
65	176	-6,86	4,14	-28,4	47,06
82	183	10,14	11,14	112,96	102,82
60	153	-11,86	-18,86	223,68	140,66
65	163	-6,86	-8,86	60,78	47,06
94	192	22,14	20,14	445,9	490,18
88	180	16,14	8,14	131,38	260,5

Use the formula for **b** to get the slope (or gradient) of the regression line:

$$b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2} = \frac{1\,308,86}{1\,610,86} = 0,812\,541\,1271... \approx 0,81$$

Substitute $(\bar{x}; \bar{y})$ and **b** into $a = \bar{y} - b\bar{x}$

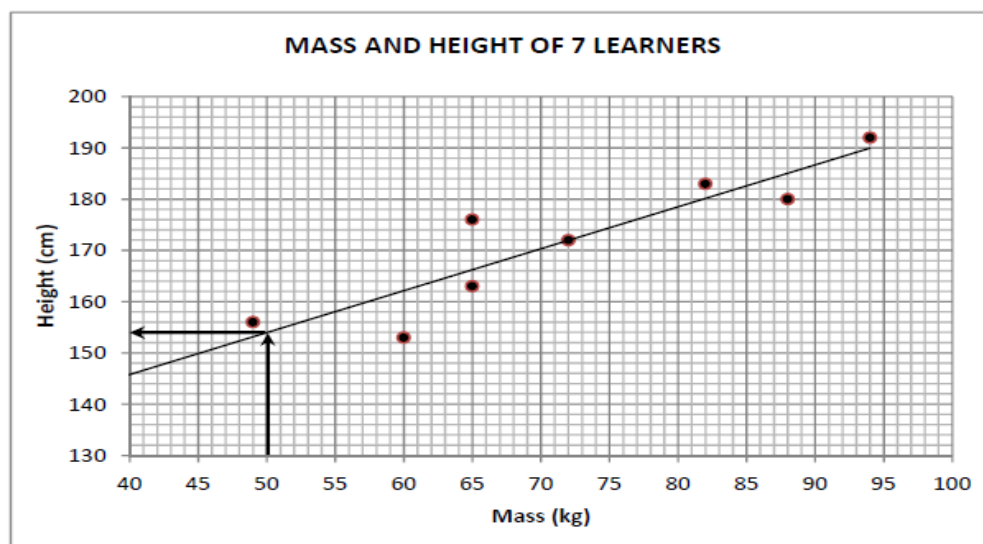
$$a = 171,86 - 71,86(0,812\,541\,1271)$$

$$= 113,4707946$$

$$\approx 113,47$$

The equation of the line of regression is therefore: $\hat{y} = 113,47 + 0,81x$

2.3, 2.4, 2.5





Activities 1&2: Some Calculations

Group organisation:	Time:	Resources:	Appendix:
Groups of 6	30 min	- Flipchart paper - Whiteboard markers	None

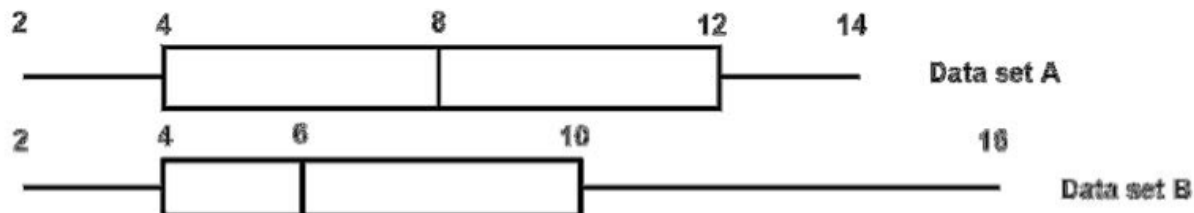
In your groups you will:

1. Select a new scribe and a spokesperson for these activities.
2. Use the flipchart and permanent markers and answer the questions as per the activity.

Activity 1: Measures of dispersions – skewed and symmetrical data

The box and whiskers diagrams of two sets, A and B, are shown below.

- 1.1 Write down what is common to both sets of data.
- 1.2 Which data set is symmetrical? State the reasons.
- 1.3 Is the other data set skewed left or right? State the reasons.



Activity 2: An integrated example

In the 2011 Household Survey, people were asked how many rooms they have in their homes. The table below shows the data taken from the North West Province:

Number of rooms per household	1	2	3	4	5	6	7	8	9
Percentage of the households	15	14	13,6	19,4	12,6	11,4	6,7	3,4	1,8

- 2.1 Draw a scatter plot to display the data.
- 2.2 Calculate the value of r , the correlation coefficient, and use it to describe the correlation that exists between the two variables.
- 2.3 Use a calculator to determine the equation of the regression line. Give your answer to TWO decimal places.
- 2.4 Draw the regression line onto your scatter plot.

Reference list

Bluman, A. (2004) *Elementary Statistics*. MacGraw Hill

Buckwell, G., (1997) *Mastering Mathematics*. Macmillan Master Series

Carter P. Et al (2007): *FOCUS in Mathematics*, Maskew Miller Longman (Pty) Ltd

Department of Basic Education (DBE) (2011): National Curriculum Statement (NCS): Curriculum and Assessment Policy Statement: Mathematics: Further Education and Training Phase: Grades 10 -12: Government Printing works. Pretoria

Department of Basic Education (DBE). (2013) National Senior Certificate Examination: National Diagnostic Report

Haighton, J. et al(2003) *Use of Maths Statistics*. Nelson Thornes Ltd.

<http://en.wikipedia.org/wiki/Wikipedia> retrieved 18 September 2014

<http://hyperphysics.phy-astr.gsu.edu/hbase/sound/tralon.html> retrieved 22 September 2014

<http://www.mathsisfun.com/> retrieved 22 September 2014

<http://www.pleacher.com/mp/mlessons/calculus/apphyper.html> retrieved 25 September 2014

<http://www.tattoodonkey.com/parabolas-in-real-life-tattoo/2/> retrieved 22 September 2014

<http://www3.ul.ie/~rynet/swconics/UP.htm> retrieved 21 September 2014

Speed B. (1997). *Higher Mathematics for GCSE*. Collins Educational.

Statistics South Africa (2010). *Census At School Results 2009*.

Watkins E., et al (2004) *Statistics in Action*. Von Hoffmann Press