GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

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SCHOOL OF LEADERSHIP & GOVERNANCE

SSIP 2017 SEPTEMBER

MATHEMATICS GRADE 12
FACILITATOR'S MANUAL

Contents

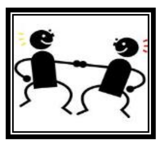
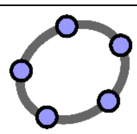
Introduction

Cognitive Levels (TIMSS)

The four cognitive levels used to guide all assessment tasks is based on those suggested in the TIMSS study of 1999. Descriptors for each level and the approximate percentages of tasks, tests and examinations which should be at each level are given below:

Cognitive levels	Description of skills to be demonstrated	Examples
Knowledge **20%**	• Estimation and appropriate rounding of numbers • Proofs of prescribed theorems and derivation of formulae • Straight recall • Identification and direct use of correct formula on the information sheet (no changing of the subject) • Use of mathematical facts • Appropriate use of mathematical vocabulary	1. Write down the domain of the function $y = f(x) = \frac{3}{x} + 2$ (Grade 10) 2. Prove that angle $A\hat{O}B$ Subtended by arc AB at the centre O of a circle is double the size of the angle $A\hat{C}B$ which the same arc subtends at the circle. (Grade 11)
Routine procedures **45%**	• Perform well known procedures • Simple applications and calculations which might involve many steps • Derivation from given information may be involved • Identification and use (after changing the subject) of correct formula • Generally, this is like those encountered in class.	1. Solve for $x: x^2 - 5x = 14$ (Grade 10) 2. Determine the general solution of the equation $2\sin(2x - 30^0) + 1 = 0$ (Grade 11)
Complex procedures **25%**	• Problems involve complex calculations and/or higher order reasoning • There is often not an obvious route to the solution • Problems need not be based on a real-world context • Could involve making significant connections between different representations • Require conceptual understanding	1. What is the average speed covered on a roundtrip to and from a destination if the average speed going to the destination is $100\ km/h$ and the average speed for the return journey is $80 km/h$? (Grade 11) 2. Differentiate $\frac{(x+2)^2}{\sqrt{x}}$ (Grade 12)
Problem solving **10%**	• Unseen, non-routine problems (which are not necessarily difficult) • Higher order understanding and processes are often involved • Might require the ability to break the problem down into its constituent parts	Suppose a piece of wire could be tied tightly around the earth at the equator. Imagine that this wire is then lengthened by exactly one metre and held so that it is still around the earth at the equator. Would a mouse be able to crawl between the wire and the earth? Why or why not? (Any grade)

Icons

1. **Discussion**	
2. **Group Activity**	
3. **Individual Activity**	
4. **Study Tips**	
5. **Notes**	
6. **Ice Breaker**	
7. **GeoGebra File**	

Training Programme

DAY ONE

Session I	
Plenary	**: 16:00 – 17: 00**
Session II	
Pre- Assessment	**: 17:00 – 18:00**
Multimedia (Identities, equations & inequalities)	**: 18:00 – 20: 00**
Dinner	**: 20:00 – 21:00**

DAY TWO

Session I	
Training Session (Algebraic Functions)	**: 08:30 – 10:00**
Tea Break	**: 10:00 – 10:15**
Session II	
Training Session (Cubic Functions)	**: 10:15 – 13:00**
Lunch Break	**: 13:00 – 14:00**
Session III	
Training Session (Cubic functions)	**: 14:00 – 15:00**
Tea Break	**: 15:00 – 15:15**
Session IV	
Training Session (Trig graphs)	**: 15: 15 – 18:00**
Tea Break	**: 18:00 – 18:15**
Session V	
Multimedia (Cognitive levels)	**: 18: 15 – 20: 00**
Dinner	**: 20:00 – 21:00**

DAY THREE

Session I	
Training Session	**: 08:30 – 10:00**
(Financial mathematics)	
Tea Break	**: 10:00 – 10:15**
Session II	
Training Session	**: 10:15 – 12:00**
(Financial Mathematics)	
Post Test	**: 12:00 – 13: 00**
Closing Session	**: 13:00 – 13:20**
Lunch	**: 13: 20 – 14: 30**

Where are we now?

Module 1	**Module 2**	**Module 3**	**Module 4**
Algebraic functions & graphs	*Cubic functions*	*Trigonometric Graphs*	*Financial Mathematics*

IDENTITIES, EQUATIONS AND INEQUALITIES

Introduction

Patterns, functions and algebra are inextricably linked. These are the tools needed in engineering, banking and a host of other industries. The study of functions and graphs are vital when creating mathematical models of real-life phenomena. Part of the aim of this section is to expose teachers to the use technology to enhance the teaching and learning of graphs and functions.

Overall aim of the module: Identities, Equations and Inequalities

The aim of the module is to distinguish between the concepts of Identities, equations and inequalities. The three concepts above are closely related but they are very different in many respects. Learners often struggle with appreciating the difference between the concepts of an identity, an equation and an inequality. A graphical approach will be used to enhance the teaching and learning process. This approach will be strengthened using technology in the form of GeoGebra.

Learning objectives of the module

After completing this topic teachers should be able to:

- Distinguish between Identities, Equations and Inequalities in terms of:

a) Concept definitions.

b) Domain of solutions for each concept.

c) Graphical representation of the solutions (done by hand).

  - Drawing of graphs represented by the statement.
  - Using interval notation.

d) Graphical representation of the solutions (done by GeoGebra).

- Correct misconceptions regarding quadratic equations and distinguishing between the forms:

a) $(x - a)(x + b) = 0$

b) $(x - a)(x + b) = c$

- Correct misconceptions regarding the use of the quadratic formula.
- Gain insights with regards to the equation of the quadratic formula.

Key Content Addressed by this module: ATP (Show curriculum mapping)

Topic	Concepts	Prior Knowledge	Methodology Teaching Strategies	Vocabulary	Assessment & Resources
Identities, Equations and Inequalities	• Definitions • Graphical representation of definitions • Use of technology to enhance teaching and learning.	• Equations	• Investigations • Discussions • Question-and answer method • Use of GeoGebra	• Identities • Equations • Inequalities • Roots • Domain • Equivalent • Shift • Sliders	• Individual activities • Group work • GeoGebra • Exam type questions

WEIGHTINGS OF CONTENT AREAS			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Algebra and Equations (and Inequalities)	30 ± 3	45 ± 3	25 ± 3
Functions and Graphs	30 ± 3	45 ± 3	35 ± 3

TERMS	**GRADE 10**		**GRADE 11**		**GRADE 12**	
	Topic	**No of weeks**	**Topic**	**No of weeks**	**Topic**	**No of weeks**
TERM 1	Equations and Inequalities	(2 Weeks)	Equations and Inequalities	(3 Weeks)	Functions and Inverses (including exponential and logarithmic)	(4 Weeks)
TERM 2	Functions	(4 Weeks)	Functions	(4 Weeks)		

Activities/ Tasks

ACTIVITY 1 **(Grade 10)** **[Duration: 5 minutes]**

Learners often struggle with appreciating the difference between the concepts of an identity, an equation and an inequality. Discuss the above concepts in your group. How can we distinguish between these concepts and how can we demonstrate these differences to the learners?

Solutions:

- An identity is an equation that is true for all values of the variable.
- An equation is true for only certain values of the variable.
  - In a linear equation, there is a maximum of one value of the variable that makes the equation true.
  - In a quadratic equation, there are a maximum of two values of the variable that makes the equation true.
  - In a cubic equation, there are a maximum of three values of the variable that makes the equation true.
- In an inequality, there may be infinitely many values of the variable that makes the inequality true.

See the GeoGebra File: Roots_of_Equation; Inequality, Parabola Roots

ACTIVITY 2 **(Grade 10; Grade 11)** **[Duration: 10 min]**

Complete the table below and **justify your choices**:

	Statement	Identity	Equation	Inequality
1.	$2^{x+3} = 64$			
2.	$2(y+3) = 2y+6$			
3.	$sin^2\theta = 1 - cos^2\theta$			
4.	$(x-3)^2 + 6x = x^2 + 9$			
5.	$sin\theta = 0.5$			
6.	$2x + 6 = 10$			
7.	$(x-3)^2 < 0$			
8.	$(x-3)^2 > 4$			

Solutions:

	Statement	Identity	Equation	Inequality
1.	$2^{x+3} = 64$		✓	
2.	$2(y+3) = 2y+6$	✓		

3.	$sin^2\theta = 1{-}cos^2\theta$	✓		
4.	$(x-3)^2 + 6x = x^2 + 9$	✓		
5.	$sin\theta = 0.5$		✓	
6.	$2x + 6 = 10$		✓	
7.	$(x-3)^2 < 0$			✓
8.	$(x-3)^2 > 4$			✓

ACTIVITY 3 **(Grade 11,12)** **[Duration: 15 min]**

Solve the following **quadratic inequalities** using:

a) algebraic manipulation
b) graphical interpretation

3.1 $x^2 + x - 6 \ \leq 0$

3.2 $x^2 + x - 6 \ > 0$

3.3 $25x^2 + 20x + 4 \ \geq 0$

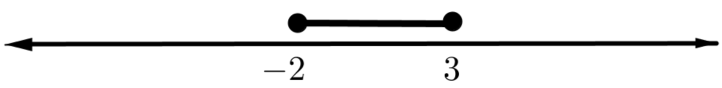
Solutions:

3.1 Algebraically:

$(x+2)(x-3) \leq 0$

$x = -2$ or $x = 3$

Interval Notation:



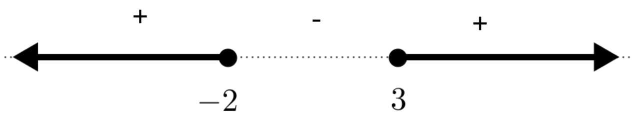
3.2 Algebraically:

$x^2 + x - 6 > 0$

$(x+2)(x-3) > 0$

Interval Notation:

The values of the expression, $x^2 + x - 6$ is shown below:



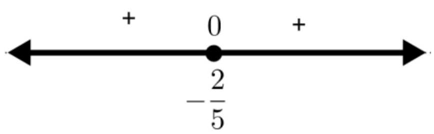
3.3 $25x^2 + 20x + 4 \geq 0$

Algebraically:

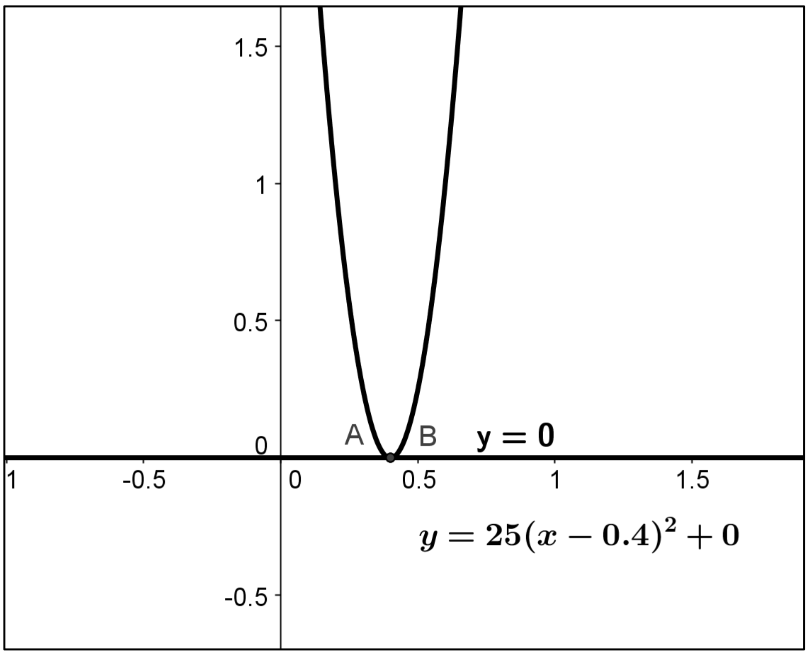
$25x^2 + 20x + 4 = (5x + 2)^2 \geq 0$

$(5x + 2)^2$ is zero for $x = -\frac{2}{5}$. For all other values of x, $(5x + 2)^2$ is positive.

Interval Notation:



Graphically:



IDENTITIES

- Note that for an **identity** such as $sin^2\theta = 1 - cos^2\theta$, the equation is true for ALL values of the variable in the given domain.
- This can be verified by substituting different values of the variable in the LHS and RHS of the equation.
- It can be shown that after some algebraic manipulation the identity reduces to $\text{f(x)} = \text{f(x)}$. This is of course true for any x.
- It can also be shown graphically that the left-hand side and the right-hand side are equivalent.

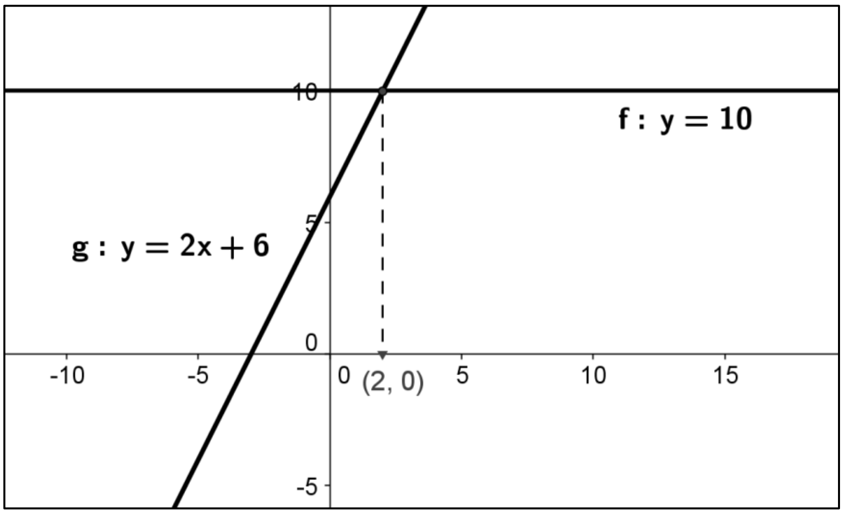
See GeoGebra File: Equivalent Trig Functions

EQUATIONS

- An **equation** is only true for some of the values of the <u>variable</u>.

<u>Example 1</u>: $2x + 6 = 10$ (This is only true when $y = 2$)

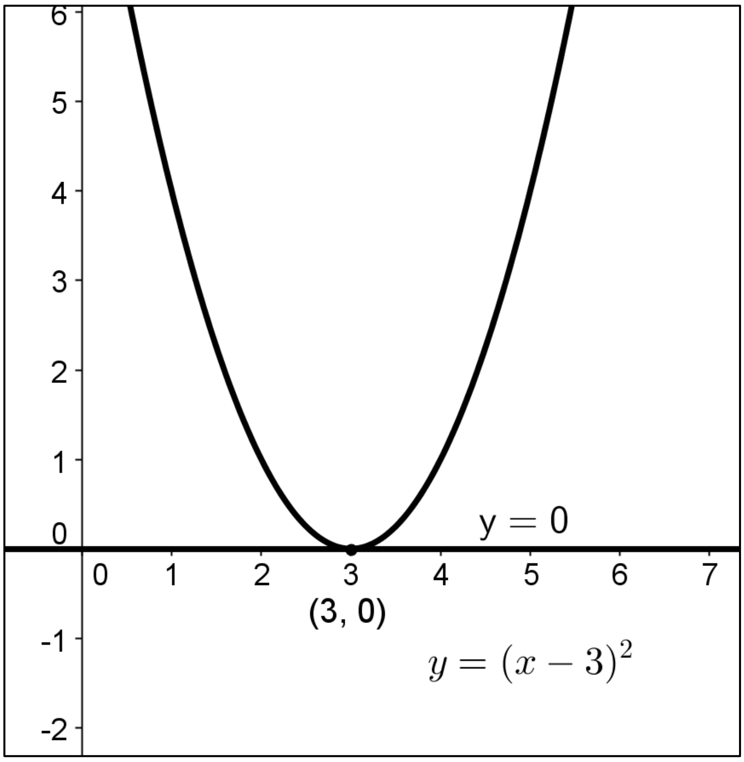
Graphical representation:



See GeoGebra File: Equation Linear

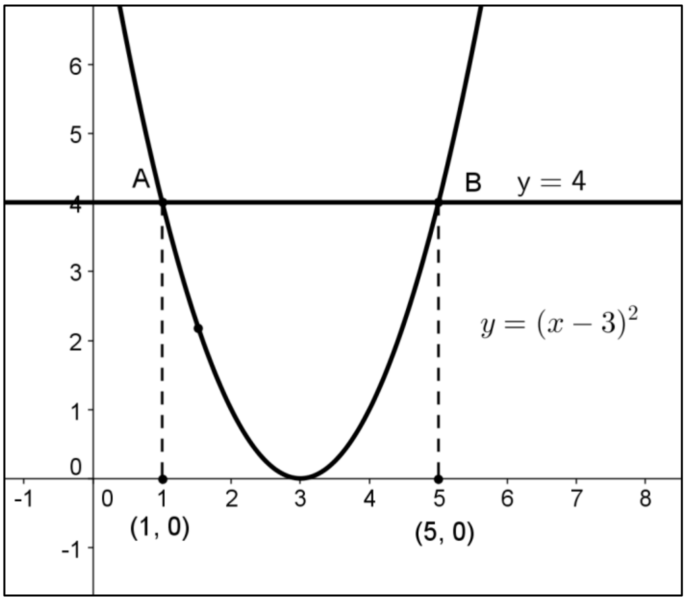
<u>Example 2</u>: $(x-3)^2 = 0$ (This is only true when $x = 3$)

Graphical representation:



<u>Example 3</u>: $(x-3)^2 = 4$ (This is true only when $x = 1$ or $x = 5$)

Graphical representation: (Note that the left-hand side of the equation can be viewed as the **graph** [$y = (x-3)^2$] and the right-hand side can be viewed as the line **graph** ($y = 4$)



See GeoGebra File: Parabola Roots

INEQUALITIES

- It is much easier to understand the **solution of an inequality** if we use graphs to represent the inequality.
- The solution of an inequality is the **domain** for which an inequality is true.
- The solution of quadratic inequalities may be read off a graph after the quadratic equation has been sketched or if the sketch is provided and the x-intercepts are known.

See GeoGebra File: Inequalities, Parabola Inverse, Parabola Roots

ACTIVITY 4 **(Grade 12)** **[Duration: 10 min]**

4.1 Consider the quadratic equation: $(x-2)(x+3)=0$ and $(x-2)(x+3)=-6$
Discuss the pitfalls and errors that learners sometimes make.

4.2 Consider the use of the quadratic formula to solve the equation below. Identify **error solutions** and offer tips and/or methods that will assist learners not to make errors in future.

<u>Question</u>:

Solve for x: $2x^2+5=7x$

Solution A:

$$x=\frac{-7\pm\sqrt{-7^2-4(2)(5)}}{2(2)}$$

Solution B:

$$x=\frac{7\pm\sqrt{(-7)^2-4(2)(5)}}{2(2)}$$

Section C:

$$x=-7\pm\frac{\sqrt{(-7)^2-4(2)(5)}}{2(2)}$$

4.3 Discuss the significance of the different expressions in the quadratic formula below:

(*See GeoGebra: Parabola QF Inverse)*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

or

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Possible responses:

4.1 $(x-2)(x+3)=0$ is in the correct form since $A.B=0 \leftrightarrow A=0\ or\ B=0$

$(x-2)(x+3)=-6$ does not imply that $(x-2)=-6$ or $(x+3)=-6$. Why?

4.2 Solution B is the only correct answer.

4.3 $x = -\frac{b}{2a}$ is the axis of symmetry;

$\frac{\sqrt{b^2-4ac}}{2a}$ is the distance from the axis of symmetry to each unequal, real root;

The value of $\sqrt{b^2 - 4ac}$ describes the nature of the roots.

2a cannot be zero. Why?

FUNCTIONS, INVERSES AND GRAPHS

Overall aim of the module: Functions and Graphs

The aim of the module is to summarise the teaching progression of different graphs over the different grades in the FET phase. A summary of the defining equations for different functions is done with a view to observe the common features as well as the effect that parameters a, p and q have on a variety of different graphs. This approach will be strengthened using technology in the form of GeoGebra to conduct investigations in this regard. In addition, the features of GeoGebra will be used to show the effect that restricting the domain of a function has on its inverse.

Learning objectives of the module

After completing this topic teachers should be able to:

- Distinguish between **shapes** of different graphs
- Understand the concepts of **domain** and **range.**
- Understand the concept of **function** and **inverse**.
- Determine **asymptotes** and their respective equations
- Investigate the **effect of parameters** on a variety of graphs
- Draw sketches of graphs and related inverse functions.
- Determine the defining equations of functions from given information.
- Determining intercepts with the axes and making deductions from given information.
- Use different forms of the defining equation of a given function or graph.

Key Content Addressed by this module: ATP (Show curriculum mapping)

Grade 10	Grade 11	Grade 12
Definition of a function. (Intuitive)	Definition of a function.	Formal definition of a function.
Point by point plotting of basic graphs such as: $y = x^2$ (Parabola) $y = \frac{1}{x}$ (Hyperbola) $y = b^x$ for $b > 0; b \neq 1$ (Exponential)	Drawing sketch graphs, determining the defining equations of functions from given information, making deductions from graphs defined by: $y = f(x) = a(x+p)^2 + q$ (Parabola) $y = f(x) = \dfrac{a}{x+p} + q$ (Hyperbola) $y = f(x) = ab^{x+p} + q$ where for $b > 0; b \neq 1$ (Exponential)	General concept of an inverse of a function. Domain restriction of a function to obtain a one-one function so that the inverse relation is a function. Determine and sketch graphs of the **inverses of functions defined by:** $y = ax + q$ (Line graph) $y = ax^2$ (Parabola)

<table>
<tr>
<td>Note: Learners must be familiar with the line graph $y = ax + q$ as well as $y = x$ (special case)</td>
<td></td>
<td>$y = b^x$
for $b > 0; b \neq 1$
(Exponential)</td>
</tr>
<tr>
<td>Concepts covered:
Shape
Domain (input values)
Range (output values)
Asymptotes
Axes of symmetry
Turning points
Intercepts on the axes</td>
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Shape
Domain (input values)
Range (output values)
Asymptotes
Axes of symmetry
Turning points
Intercepts on the axes</td>
<td>Concepts covered:
Shape and symmetry
Domain (input values)
Range (output values)
Asymptotes
Axes of symmetry
Turning points
Intercepts on the axes
Maxima and minima
Intervals on which increasing/decreasing
Average gradient (rate of change)</td>
</tr>
<tr>
<td>Investigate the effect of a and q on the graphs defined by:
$y = a.f(x) + q$</td>
<td>Revise the effect of a and q on the graphs in Grade 10.
Investigate the effect of p on the graphs of the functions defined by:
$y = f(x) = a(x + p)^2 + q$
(Parabola)
$y = f(x) = \frac{a}{x + p} + q$
(Hyperbola)
$y = f(x) = ab^{x+p} + q$
where for $b > 0; b \neq 1$
(Exponential)
Investigate (numerically) the concept of average gradients between two points on a curve leading to an intuitive understanding of the concept of the gradient of a curve at a point.</td>
<td></td>
</tr>
</table>

Topic	Concepts	Prior Knowledge	Methodology Teaching Strategies	Vocabulary	Assessment & Resources
Functions, Inverses and Graphs	• Graph sketching • Shape, domain and range • Function and inverse • Asymptotes • Effect of parameters	• Point by point Plotting • Intuitive definition of functions	• Investigations • Discussions • Question-and answer method • GeoGebra	• Functions • Inverses • Domain • Range • Increasing/decreasing function • Asymptotes • Parameters • Vertical line test	• Individual activities • Group work • GeoGebra • Exam type questions

FUNCTIONS AND GRAPHS

Activities/ Tasks

Summary of the equations for different functions

THE LINE GRAPH

The Line Graph **(Grade 9-12)**	A. Standard form: $\boxed{y = mx + c}$ (where $\boldsymbol{m}$ represents the **gradient** of the line graph and $\boldsymbol{c}$ represents the $\boldsymbol{y}$**-intercept**) B. Gradient and point form: $\boxed{y - y_1 = m(x - x_1)}$ C. Two points form: $\boxed{\frac{y_2 - y_1}{x_2 - x_1} = m}$

ACTIVITY 1 **(Grade 10)** **[Duration: 10 min]**

Given: $f: 3y - 2x - 1 = 0$, write down:

1.1 An equation of a line that is parallel to f and passes through the point $H(-3;-1)$.

1.2 Is the point (5;4) on f? Explain.

1.3 What is the value of the x -intercept of f?

1.4 Write down any two other points that lie on f. Calculate the average gradient between these points. What do you notice?

Solutions:

1.1 $3y - 2x - 1 = 0 \rightarrow y = \frac{2}{3}x + \frac{1}{3}$

$$y = ax + q \rightarrow y = \frac{2}{3}x + q$$

Substitute $(-3;-1)$: $-1 = \frac{2}{3}(-3) + q \rightarrow q = 1$

Therefore, the equation: $y = \frac{2}{3}x + 1$

1.2 Substitute $(5;4)$ in LHS and RHS of $y = \frac{2}{3}x + \frac{1}{3}$

LHS= 4; RHS= $\frac{2}{3}(5) + \frac{1}{3} = \frac{11}{3} \therefore LHS \neq RHS$, $(5;4)$ is not on f.

1.3 x- intercept $(y = 0)$, $3(0) - 2x - 1 = 0 \rightarrow 2x = -1 \rightarrow x = -\frac{1}{2}$

1.4 The average gradient between any two points on the line are equal.

THE PARABOLA

The Parabola	A. Standard form: $\boxed{y = ax^2 + bx + c \text{ (where } a \neq 0)}$ B. x-intercept (root) form: $\boxed{y = a(x - r_1)(x - r_2)}$ C. Turning Point form: $\boxed{y = a(x + p)^2 + q}$

THE EXPONENTIAL GRAPH

The Exponential Graph	**Standard form:** $\boxed{y = f(x) = ab^{x+p} + q}$ where for $b > 0; b \neq 1$ **Horizontal asymptotes**: $ab^{x+p} > 0$ for all x. Why? $ab^{x+p} + q > 0 + q$ This means that: $y = f(x) = ab^{x+p} + q > q$ So $y = q$ is the horizontal asymptote for the exponential graph.

THE HYPERBOLA

<table>
<tr>
<td>The Hyperbola

See the GeoGebra File: Hyperbola general</td>
<td>Standard form:

$$y = \frac{a}{x+p} + q$$

Alternative form:

$$\boxed{(y-q) = \frac{a}{(x+p)}}$$

or

$$\boxed{(x+p) = \frac{a}{(y-q)}}$$

Clearly, the vertical asymptote is given by:

$$\boxed{x = -p}$$

$$x + p = 0 \leftrightarrow x = -p$$

And the horizontal asymptote is given by:

$$\boxed{y = q}$$

$$y - q = 0 \leftrightarrow y = q$$
</td>
</tr>
</table>

ACTIVITY 2 **(Grade 10)** **[Duration: 5 min]**

Show that the equation, $y = \frac{a}{x+p} + q$, can be written in the alternative forms indicated above. How did we arrive at the conclusion that, $x = -p$ and $y = q$, are the vertical and horizontal asymptotes respectively?

Solutions:

In the form,

$$(y-q) = \frac{a}{(x+p)}$$

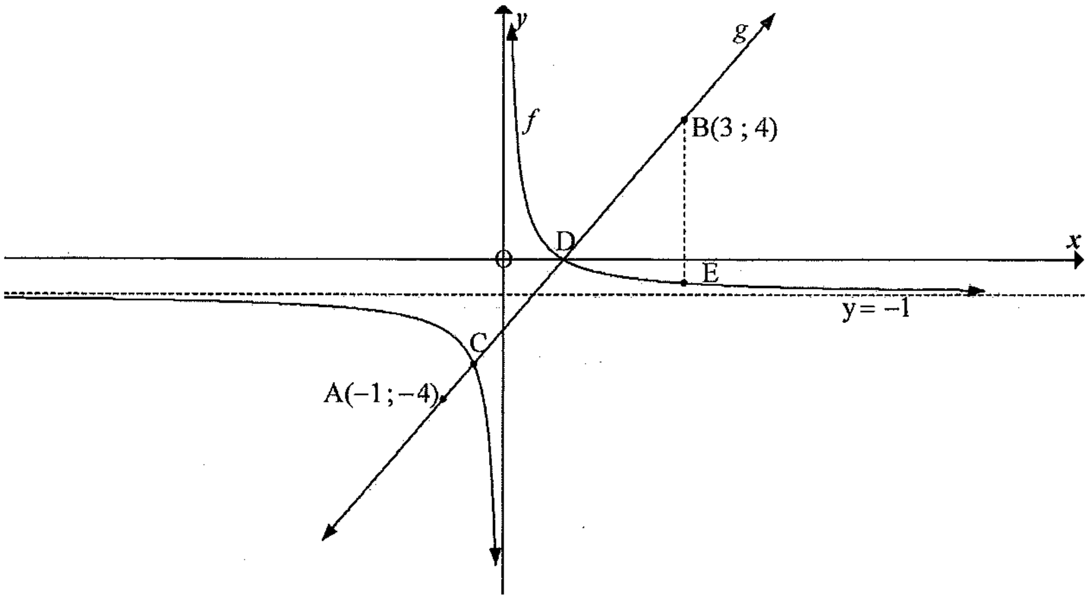
, it is clear that $x + p \neq 0$. This means that $x \neq -p$ for any x.

Similarly, in the form,

$$(x+p) = \frac{a}{(y-q)}$$

it is clear that $(y-q) \neq 0$. This in implies that $y \neq q$ for any x. We have just proved that $x = p$ and $y = q$ are the vertical and horizontal asymptotes of the hyperbolic graph respectively.

ACTIVITY 3 **(Grade 10-12)** **[Duration: 15 min]**



Consider the graphs $f(x) = \frac{1}{x} - 1$ and $g(x) = ax + q$ in the diagram above. Points $A(-1; -4)$ and $B(3; 4)$ lie on graph g. The graphs intersect at points C and D. BE is drawn parallel to the y-axis, with E on f.

3.1 Determine the asymptotes of f.

3.2 For which value(s) of x will $f(x) = g(x)$?

3.3 For which values of x will $f(x) \geq g(x)$?

3.4 Determine the length of BE.

Solutions:

3.1 $(y+1) = \frac{1}{(x-0)} \leftrightarrow y = -1$ and $x = 0$ (y-axis) are the horizontal and vertical asymptotes respectively.

OR

Interpret from the graph.

3.2 Use two points to find the equation of $g(x)$.

$a = 2$, $q = 2$. Therefore, the equation of $g(x) = 2x - 2$

The two graphs have the same y −intercepts when:

$\frac{1}{x} - 1 = 2x - 2 \leftrightarrow 2x^2 - x - 1 = 0 \leftrightarrow (2x+1)(x-1) = 0 \leftrightarrow x = -\frac{1}{2}$ or $x = 1$

3.3 $f(x) \geq g(x)$ for $x \leq -\frac{1}{2}$ or $0 < x \leq 1$

3.4 Let $E(3; t)$ why?

Since E is on f, $t = \frac{1}{3} - 1 \leftrightarrow t = -\frac{2}{3}$

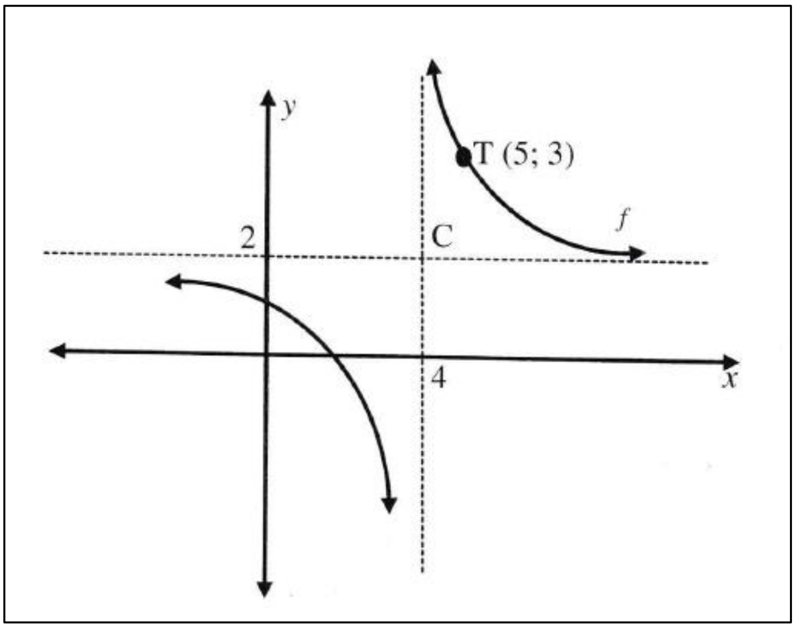
Length of BE $= 4 + \frac{2}{3} = 4\frac{2}{3}$

ACTIVITY 4 **(Grades 10-12)** **[Duration: 15 min]**

In the sketch below, the graph of:

$$f(x) = \frac{a}{x+p} + q$$

has asymptotes $x = 4$ and $y = 2$. The point $T(5; 3)$ is a point on f and C is the point of intersection of the asymptotes.



4.1 Determine the values of a, p and q.

4.2 Give the equation of h, the resulting graph after f is shifted vertically upward by 3 units.

4.3 If the graph of f is symmetrical about the line $y = -x + c$, determine the value of c.

Solutions:

4.1 From the sketch, the asymptotes are $y = 2$ and $x = 4$. This implies that **on f**, $y \neq 2 \rightarrow y - 2 \neq 0$ and $x \neq 4 \rightarrow x - 4 \neq 0$. Thus, the function f is defined by:

$$y - 2 = \frac{a}{x - 4}$$

Substitute T(5;3) in f to determine the value of a.

$$3 - 2 = \frac{a}{5 - 4} \rightarrow a = 1$$

The defining equation for f :

$$y - 2 = \frac{1}{x - 4}$$

Or in standard form:

$$y = \frac{1}{x - 4} + 2$$

Therefore: $a = 1, p = -4$ and $q = 2$

4.2 $h(x) = f(x) + 3$

$$h(x) = (\frac{1}{x - 4} + 2) + 3 = \frac{1}{x - 4} + 5$$

4.3 $y = -x + c$

The point (4;2) is on the line of symmetry. Why?

Substitute in: $2 = -4 + c \rightarrow c = 6$

ACTIVITY 5 **(Grade 11)** **[Duration: 10 min]**

$J(x) = \frac{1}{2}b^{x+p} + 3$ passes through $A(2; 3\frac{1}{2})$ and $B(3; 5)$.

5.1 What type of graph is defined by J?

5.2 What is the equation of the asymptote?

5.3 Determine the values of b and p.

5.4 Determine the value of the y-intercept of J?

Solutions:

5.1 Exponential graph

5.2 $y = 2$

5.3 Substitute $A(2;3\frac{1}{2})$ and $B(3;5)$ into $y=\frac{1}{2}b^{x+p}+3$

$A\left(2;3\frac{1}{2}\right)\rightarrow 3\frac{1}{2}=\frac{1}{2}b^{2+p}+3\rightarrow\frac{1}{2}=\frac{1}{2}b^{2+p}\rightarrow 2+p=0\rightarrow p=-2$

$A(3;5)\rightarrow 5=\frac{1}{2}b^{3-2}+3\rightarrow 2=\frac{1}{2}b\rightarrow b=4$

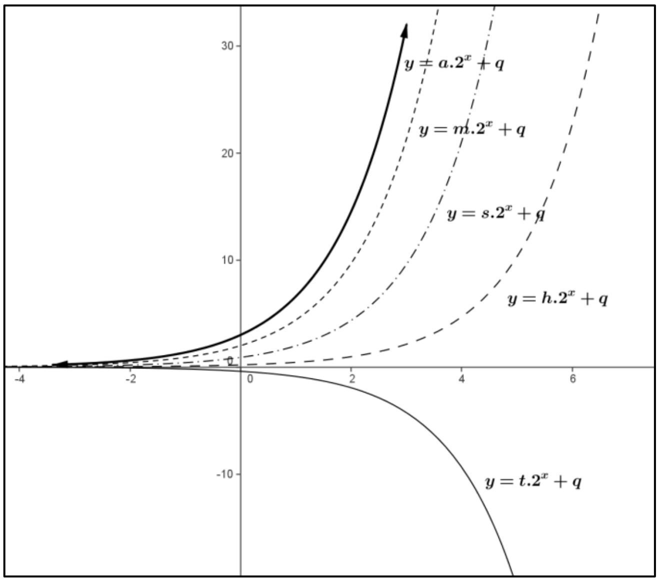
5.4 $J(x)=\frac{1}{2}.\ 4^{x-2}+3$

$x=0\rightarrow y=\frac{1}{2}.\ 4^{0-2}+3=\frac{1}{32}+3=\frac{97}{32}$

ACTIVITY 6 **(Grade 10-12)** **[Duration: 10 min]**

INVESTIGATION 1

Use the sketch below and your knowledge of the exponential function to make deductions about the parameters. (Note that q and b are kept constant)



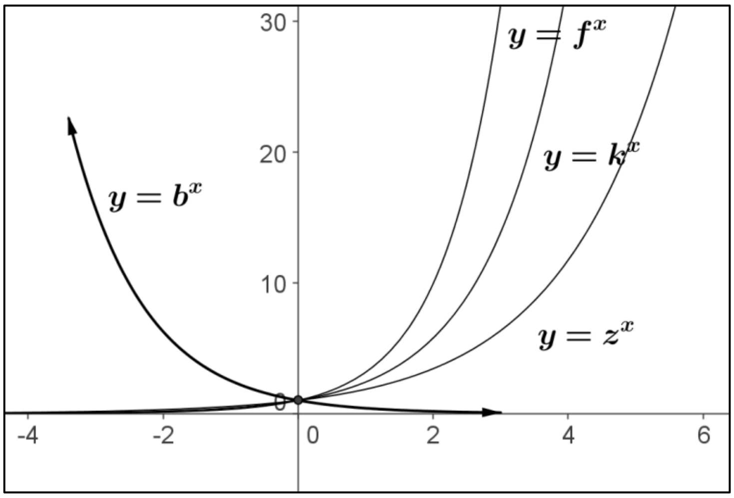
Possible Responses:

1. The shape changed
2. $a>m>s>h>t$
3. $t<0$
4. $q=0$, asymptote is $y=0$

See GeoGebra File: Param a b q exp

INVESTIGATION 2

Investigate the relationship between the change in the same parameter. Note that a and q are kept constant. This in an open-ended investigation so each group must explore on their own and report on what changed and what stayed the same. Offer reasons for your observations.

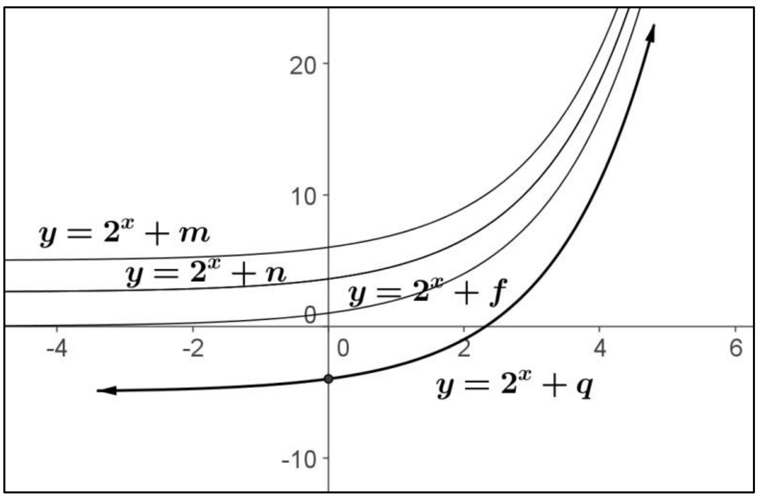


Possible Responses:

1. The shape changed
2. $0 < b < 1$
3. $f > k > z > 1$
4. The y-intecept did not change
5. The asymptote is $y = 0$

INVESTIGATION 3

Investigate the relationship between the change in the same parameter. Note that a and b are kept constant. This in an open-ended investigation so each group must explore on their own and report on what changed and what stayed the same. Offer reasons for your observations.



Possible Responses:

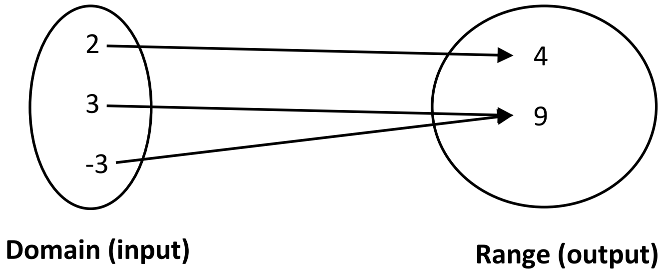
1. The shape remained the same.
2. The graph shifted upward/downward as 'q' increased or decreased.
3. These are not the y-intecepts of the exponential function.

FUNCTIONS AND INVERSES

(Grade 12 Only)

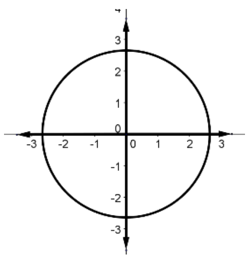
A function

- is a rule where **each input value** has a **single output value.**
- maps each element of one set A to a unique element in another set B
- is a rule by which each element in the domain is associated with only one element of the **range**.

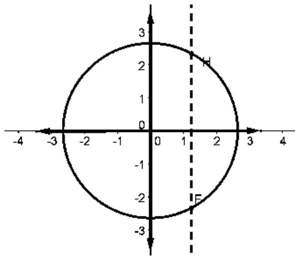
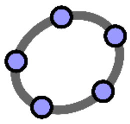


ACTIVITY 1 **(Grade 10)** **[Duration: 5 min]**

Classify the relations by completing the table (Please tick ✓)

	Relation	Not a function	Function	One-one function	Many-to-one function
1.	$y = sin\theta$				
2.	$\{(4;-1);(0;4);(1;4);(3;7)\}$				
3.					

Solution:

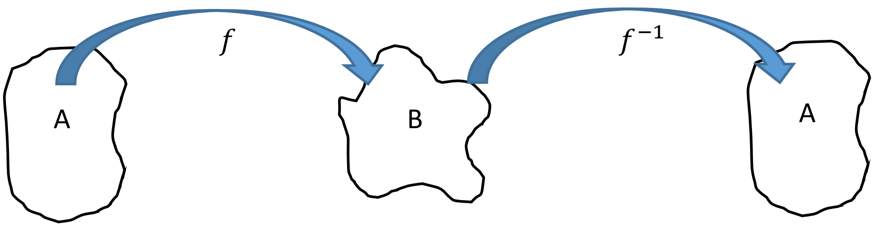
	Relation	Not a function	Function	One-one function	Many-to-one function
1.	$y = sin\theta$		✓		✓
2.	$\{(4; -1); (0; 4); (1; 4); (3; 7)\}$		✓		✓
3.		✓	 *See GeoGebra File: Circle_Vertical_line_Test*		

DEFINITIONS

- If a function f maps each element in a set A to another element in set B, then the inverse of f denoted by f^{-1} is another function that maps each element in B back to the original element in set A.
- Note that:

$$f^{-1} \neq \frac{1}{f}$$

- f and f^{-1} and generally different functions but are related as shown below.
- Note that for the inverse (f^{-1}) to be a function, we may need to restrict the domain of f to become a one-one function. Discuss.
- Graphically,



ACTIVITY 2 **(Grade 12)** **[Duration: 8 min]**

Using the above information, complete the statements below:

2.1 For any $x \in A, f(x)$ will be an element of set ______

2.2 $f(x)$ is an element of set ________ for all x in set ________

2.3 $f\left(f^{-1}(x)\right) =$ _________

2.4 $f^{-1}\left(f(x)\right) =$_________

2.5 $f^{-1}\left(f(x)\right) \in$ _________

Solutions:

2.1 For any $x \in A, f(x)$ will be an element of set **B**

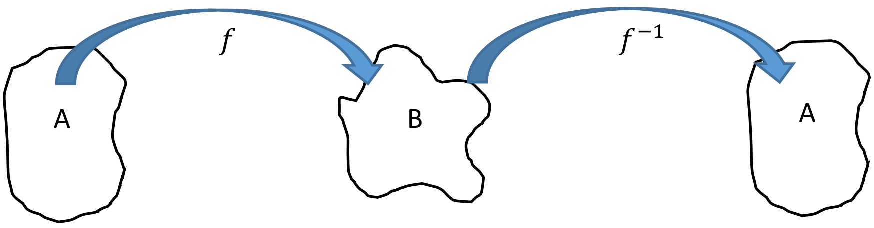
2.2 $f(x)$ is an element of set **B** for all x in set **A**

2.3 $f\left(f^{-1}(x)\right) = \boldsymbol{x}$

2.4 $f^{-1}\left(f(x)\right) = \boldsymbol{x}$

2.5 $f^{-1}\left(f(x)\right) \in$ **A**

INTERCHANGE OF VARIABLES



- Note that the elements in set B constitute the image of f (that maps the input elements from set A)
- These elements now become the input elements for function f^{-1}.
- **We can therefore say that the $\boldsymbol{y}$ is replaced by $\boldsymbol{x}$**
- In a very simplistic way, we can say the f and f^{-1} does the exact opposite/reverse to the input values.

ACTIVITY 3 **(Grade 12)** **[Duration: 10 min]**

INVESTIGATION:

In the table below, each ordered pair is transformed to its image by a function.

3.1 Complete the table below:

Rule: f	Rule: f^{-1}
A (1;2)	A' (___; ____)
B (2;5)	B' (___; ____)
C (4;11)	C' (___; ____)
D (-2; -7)	D' (___; ____)

3.2 Plot the points on a Cartesian plane.

3.3 Determine the equation of the axis of symmetry

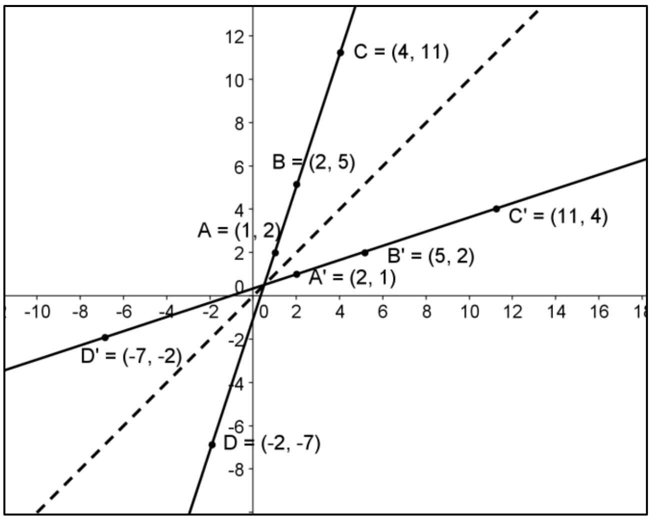
3.4 What do you notice?

Solutions:

3.1

Rule: f	Rule: f^{-1}
A (1;2)	A' (2;1)
B (2;5)	B' (5;2)
C (4;11)	C' (11;4)
D (-2; -7)	D' (-7; -2)

3.2



3.3 $y = x$

3.4 The points are reflected about the line $y = x$. This means that, to obtain the inverse function, the original function must be reflected about the line: $y = x$

STEPS TO FOLLOW WHEN DETERMINING THE INVERSE

To find the equation of the inverse of a function, follow the steps below:

- Write the original function in the form: $y =$…
- Interchange x and y
- Make y the subject.

ACTIVITY 4 **(Grade 12)** **[Duration: 15 min]**

Determine the equations of the inverse functions for each of the following:

4.1 $y = -3x + 10$

4.2 $y = 2^x$

4.3.1 $y = 2x^2$

4.3.2 For which **restricted domain** would the inverse relation be a function in 4.3.1. Explain with the use of a sketch.

4.4.1 If $g(x) = 3.2^{x+3}$ and $f(x) = -x + 3$, determine $g(f^{-1}(1))$.

4.4.2 What do you notice about f and f^{-1}?

Solutions:

4.1 Exchange x and y: $x = -3y + 10$

Make y the subject: $y = -\frac{1}{3}x + \frac{10}{3}$

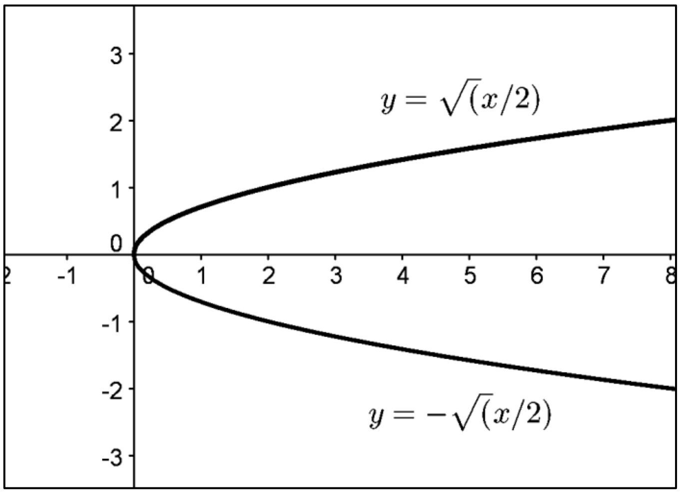
4.2 Exchange x and y: $x = 2^y$

Make y the subject: $y = log_2 x$

4.3.1 Exchange x and y: $x = 2y^2$

Make y the subject: $y = \mp\sqrt{\frac{1}{2}x}$

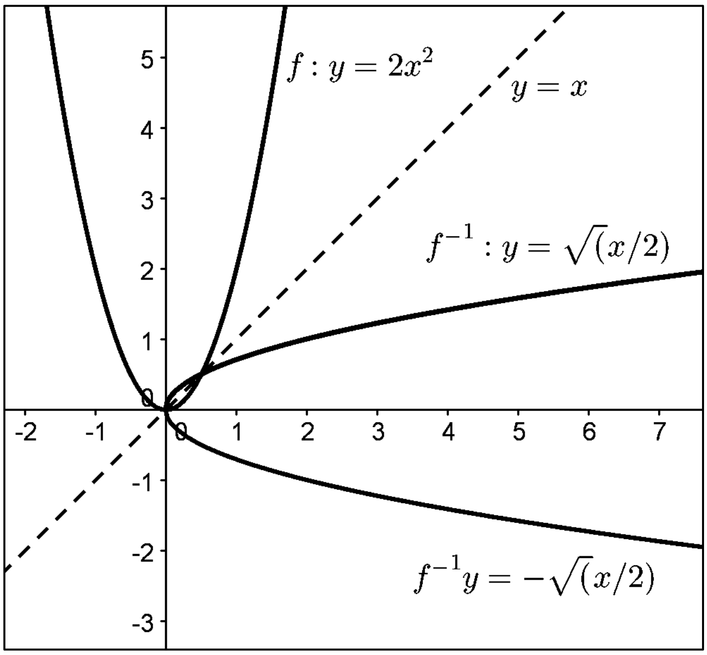
4.3.2 It can be seen in the sketch below that the inverse relation is not a function. Why?



There are two ways in which we can restrict the domain such that the inverse relation is a function.

If the domain is restricted to $x \leq 0$ or $x \geq 0$ (See the sketch below)

See GeoGebra File: Parabola_inverse_domain_restriction



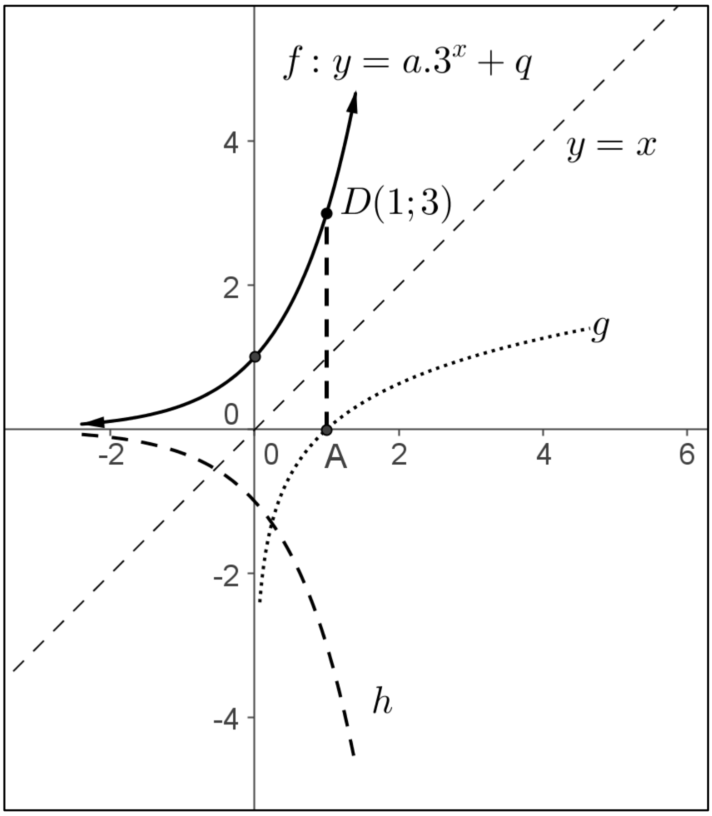
4.4.1 $\quad f^{-1}(x) = -x + 3 \leftrightarrow f^{-1}(1) = -(1) + 3 = 2$

$$g\left(f^{-1}(1)\right) = g(2) = 3.2^{2+3} = 96$$

4.4.2 $\quad f$ and f^{-1} are the same function (in this case).

ACTIVITY 5

5. Study the sketch below and answer the questions that follow:



5.1 What is the range of the function defined by f.
5.2 Determine the values of a and q.
5.3 Determine the defining equation of g, the inverse of $y = a.3^x + q$
5.4 Determine the defining equation of h
5.5 What is the domain of g.
5.6 Determine the length of AD

Solutions:

5.1 Domain: $y > 0$ for all real values of x or $y \in (0; \infty)$
5.2 $q = 0$ (asymptote $y = 0$)
Substitute (1;3): $3 = a.3^1 \leftrightarrow a = 1$
5.3 $g: y = log_3 x$
5.4 $y \to -y \therefore -y = 3^x \to y = -3^x$
5.5 $x > 0$ or $x \in (0; \infty)$
5.6 $A(1;0)$ – See the graph. $AD = 3\ units$

Summary Big Ideas

- A graphical representation of an equation does not only enhance the understanding of the concept but leads to greater insights with respect to inequalities and the nature of the roots.
- Re-writing standard form equations such as the equation for the hyperbola ($y = \frac{a}{x+p} + q \leftrightarrow y - q = \frac{a}{x+p} \leftrightarrow x + p = \frac{a}{y-q}$) creates consistent teaching with an understanding. This leads to greater degrees of retention by learners and teachers alike.
- Some misconceptions, relating to quadratic functions and the use of the quadratic formula, are identified and solutions are offered.
- The use of technology greatly assists in teaching inverse functions. In addition, the restrictions on the domain of a function f such that the inverse relation is also a function is clearly illustrated using GeoGebra.

ICT Integration

IWB/Multimedia/Web 2.0 Tools/Web Resources/Triggered PPT/GeoGebra

Conclusion

This section covered some of the topics relating to Patterns, Functions and Algebra. Rather than attempting to exhaust the topic, more attention was given to the methodology of teaching using technology and good mathematical insights. The GeoGebra environment also lends itself to the investigative approach as done in this section.

Where are we now?

Module 1	Module 2	Module 3	Module 4
Algebraic functions & graphs	*Cubic functions*	*Trigonometric Graphs*	*Financial Mathematics*

CUBIC FUNCTIONS

Introduction

We study algebraic expressions $y = f(x)$ whose highest power is 3. For instance, $y = f(x) = x^3$. Thus, these are expressions of the form $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$, where a, b, c and d are real numbers, i.e., any or all of b, c and d can be zero. Examples of cubic equations include, $x^3 - 6x^2 + 11x - 6 = 0$, $4x^3 + 57 = 0$, $x^3 + 9x = 0$

Overall aim of the module: Cubic Functions

To introduce or reintroduce teachers to polynomial functions 2) To define polynomial and to provide teachers with a standard form of a polynomial 3) To show teachers the different characteristics of cubic graphs 4) Teach teachers how to graph different polynomials and how to model/interpret different cubic graphs.

Learning objectives of the module

After completing this topic teachers should be able to: know how best cubic functions can be taught to learners, especially with regard to

a) Their factorization

b) Graphs associated with them

c) Characteristics of cubic graphs

d) Applications of maxima and minima in real-life problems

- Determining characteristics of functions. i.e, expose learners to all aspects of functions including:
  - ✓ Where a function is increasing or decreasing;
  - ✓ The stationery points of a function;
  - ✓ Local maxima and minima of the function;
  - ✓ The concept of concavity;
    - ❖ Concave up;
    - ❖ Concave down;
  - ✓ Where a function is concave or convex;
  - ✓ The concept of inflection point of a function;
  - ✓ Interpretation of the equation and the graph;
  - ✓ Finding the equation given information and transformations;
  - ✓ Roots of equations or zeros of functions

- ✓ Points of intersection;
- ✓ Intervals where graphs are relative to one another under given conditions;
- ✓ Gradients and equations of tangents;
- ✓ Sketch the graph of a function;

- The difference between the function $f(x)$ and its gradient $f'(x)$

Key Content Addressed by this module: ATP (Show curriculum mapping)

Topic	Concepts	Prior Knowledge	Methodology Teaching Strategies	Vocabulary	Assessment & Resources
Calculus	• Graph sketching • Concavity • Points of inflection • tangents	Definition of functions	• Investigations • Discussions • Question-and answer method	• Functions; • Concavity' • Point of inflections	• Individual activities • Exam type questions

WEIGHTINGS OF CONTENT AREAS			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Differential Calculus			35 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3

TERMS	**GRADE 10**		**GRADE 11**		**GRADE 12**	
	Topic	**No of weeks**	**Topic**	**No of weeks**	**Topic**	**No of weeks**
TERM 1	Trigonometry	(3 Weeks)			Trigonometry	(2 Weeks)
TERM 2	Trig. Functions	(1 Week)	Trig Graphs and Equations included	(4 Weeks)	Trigonometry	(2 Weeks)
TERM 3	Trigonometry	(2 Weeks)	Sine, Cosine and Area Rules	(2 Week)	Functions (Polynomials) and Differential Calculus	(1 Week) (3 Weeks)

Activities/ Tasks

A. INTERCEPTS WITH THE AXES

- ✓ For the *y*-intercept, let $x = 0$ and solve for y (i.e., $y = d$). Intercept at (0; d)
- ✓ For the *x*-intercepts, let $y = 0$ and solve for x. there could be one, two or three real roots. Use:
  1. Common factor
  2. Grouping
  3. Factor theorem (Only if (i) and (ii) are not applicable);

For example:

Solve for *x*: $f(x)=x^3-6x^2+11x-6$

$x^3-6x^2+11x-6=0$

$(x-1)(x-2)(x-3)=0$

Solutions: There are three real roots, all different. The solutions are $x=1$, $x=2$ and $x=3$


Activity

Solve for *x*:

1. $f(x)=x^3-5x^2+8x-4$
2. $f(x)=x^3-3x^2+3x-1$
3. $f(x)=x^3+x^2+x-3$

Solutions:

1.

$x^3-6x^2+11x-6=0$

$(x-1)(x-2)^2=0$

$x=1 \quad and \quad x=2$

We do have three real roots but two of them are the same because of the term $(x-2)^2$

We only have two distinct solutions.

2. $f(x)=x^3-3x^2+3x-1$

$$x^3 - 3x^2 + 3x - 1 = 0$$
$$(x-1)^3 = 0$$
$$\therefore x = 1$$

Although there are three factors, they are the same and we only have a single solution $x = 1$

3. $f(x) = x^3 - 3x^2 + 3x - 1$

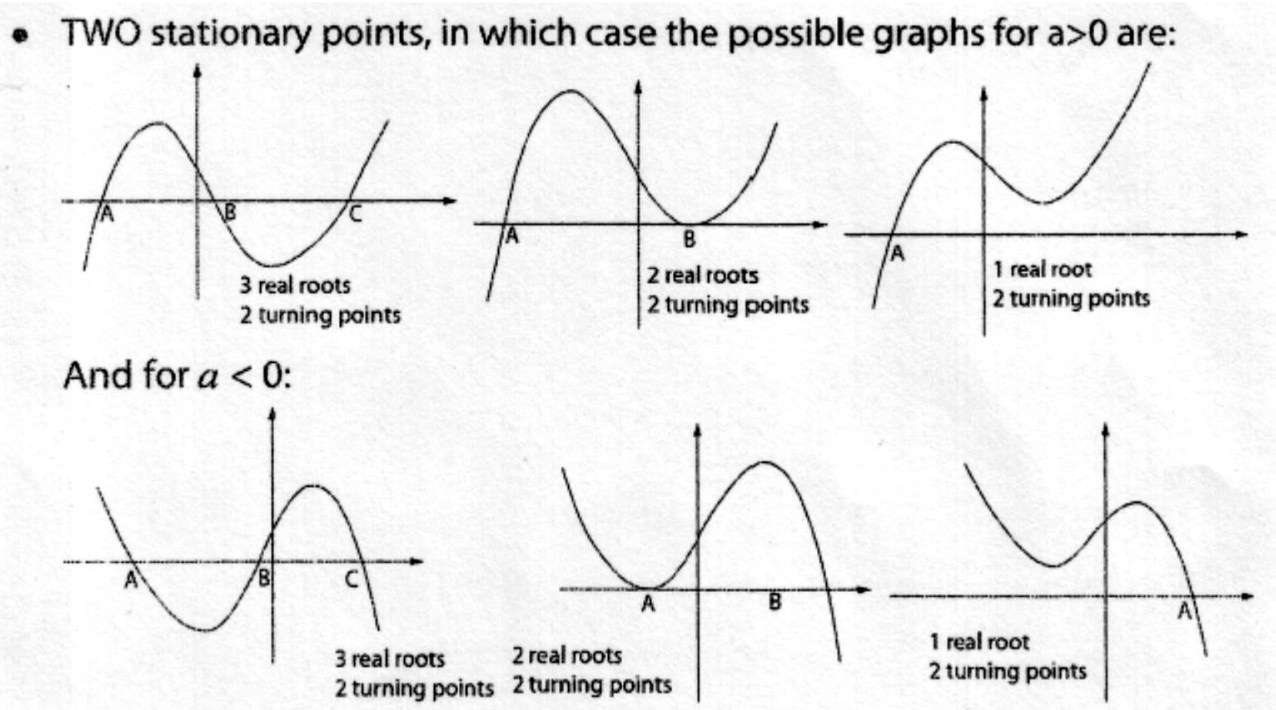
$$x^3 + x^2 + x - 3 = 0$$
$$(x-1)(x^2 + 2x + 3) = 0$$

The quadratic $x^2 + 2x + 3 = 0$ has no real solutions, so $x = 1$ is the only solution

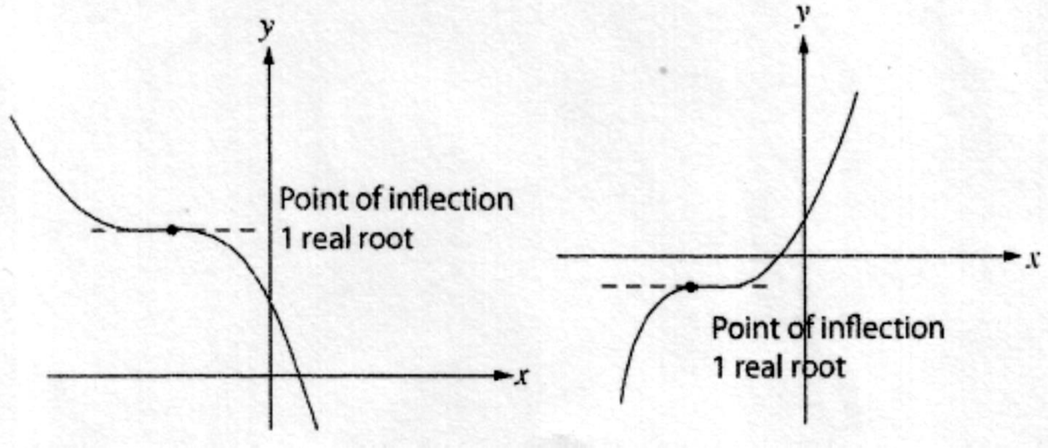
a) Cubic should always be re-arranged into its standard form;
b) Use **factor theorem** to determine the first factor;
c) Find the other factors by using **synthetic division**

B. STATIONERY POINTS (LOCAL MAXIMUM & LOCAL MINIMUM)

Def: Stationery (or ***critical***) point of a function *f*(*x*) is any point *x* in the domain of *f*(*x*) where $f'(x) = 0$, or where $f'(x)$ does not exist.



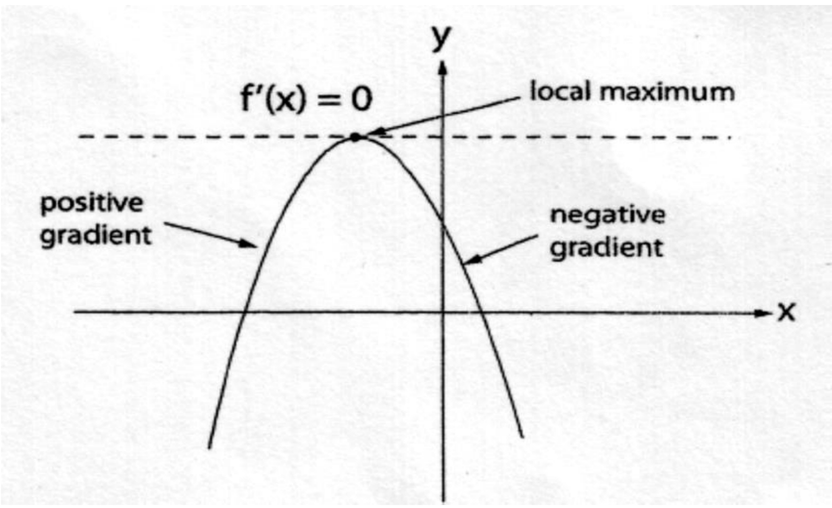
- ONE stationary point, in which case the possible graphs are:



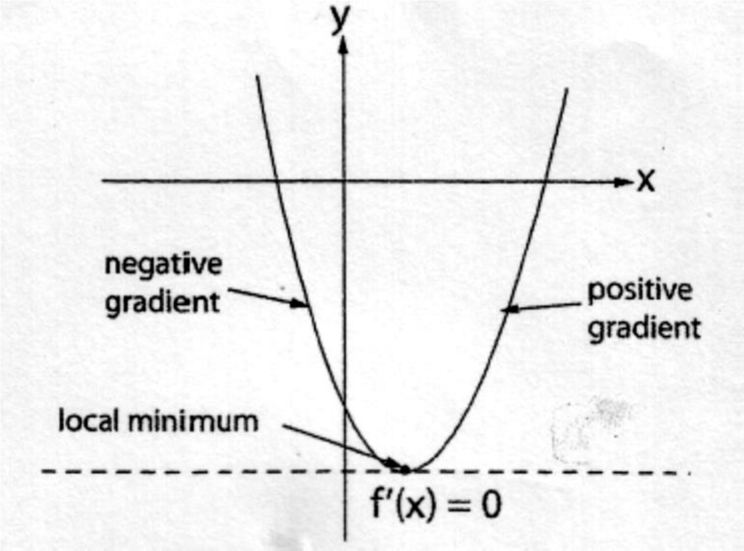
- Determine the derivative
- Solve for x: $f'(x) = 0$
- Determine the y- coordinate (s) of the turning point through **substitution into original equation**.
- If the function has two stationery points, establish whether they are maximum or minimum turning points;

There are three kinds of stationery points:

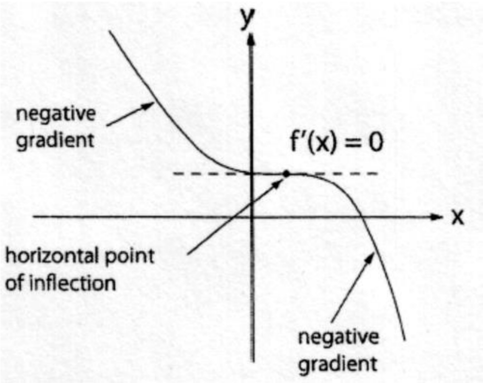
1. Local maximum



2. **Local minimum**



3. **Point of inflection**



Not all points of inflection are stationery points. The gradient of the curve <u>must be zero</u> at the stationery point.

Example:

Determine the stationery points of the following:

$f(x)=x^3-x^2-8x+12$

Solution:

$$f(x)=x^3-x^2-8x+12$$
$$f'(x)=3x^2-2x-8$$
$$0=(3x+4)(x-2)$$
$$\therefore\ x=-\frac{4}{3}\qquad or\qquad x=2$$
$$\therefore f(-\frac{4}{3})=(-\frac{4}{3})^3-(-\frac{4}{3})^2-8(-\frac{4}{3})+12=18.5\ \Rightarrow(-\frac{4}{3};18{,}5)$$
$$f(2)=(2)^3-(2)^2-8(2)+12=0\qquad\Rightarrow(2;0)$$

Determine the stationery points of the following:

1. $f(x)=x^3+3x^2$
2. $f(x)=-x^3+12x^2+16$
3. $f(x)=2x^3-11x^2+12x$

Solutions:

1. $f(x)=x^3+3x^2$

   $f'(x)=3x^2+6x$ $\qquad [\,find\ \ f'(x)]$

   $0=3x^2+6x$ $\qquad [at\ turning\ point,\ f'(x)=0]$

   $3x(x+2)=0$

   $\therefore 3x=0\qquad or\qquad x+2=0$

   $\therefore x=0\ \ or\qquad x=-2$

   $\therefore\ f(0)=(0)^3+3(0)^2=0\qquad\Rightarrow(0;0)\qquad [Substt\ \text{into}\ original\ eqn]$

   $\therefore\ f(-2)=(-2)^3+3(-2)^2=4\qquad\Rightarrow(-2;4)$

2. $f(x)=-x^3+12x^2+16$

   $f'(x)=-3x^2+12$ $\qquad [\,find\ \ f'(x)]$

   $0=-3x^2+12$ $\qquad [at\ turning\ point,\ f'(x)=0]$

   $-3(x^2-4)=0$

   $-3(x-2)(x+2)=0$

   $\therefore x=2\ \ or\qquad x=-2$

   $\therefore\ f(2)=-(2)^3+12(2)^2+16=32\qquad\Rightarrow(2;32)\qquad [Substt\ \text{into}\ original\ eqn]$

   $\therefore\ f(-2)=(-2)^3+12(-2)^2+16=0\qquad\Rightarrow(-2;0)$

3. $f(x)=2x^3-11x^2+12x$

$f^{'}(x)=6x^2-22x+12$ $\qquad$ $[find\ \ f^{'}(x)]$

$0=6x^2-22x+12$ $\qquad$ $[at\ turning\ po\text{int},\ f^{'}(x)=0]$

$(3x-2)(x-3)=0$

$\therefore x=\dfrac{2}{3}\ or \qquad x=3$

$\therefore\ f(\dfrac{2}{3})=2(\dfrac{2}{3})^3-11(\dfrac{2}{3})^2+12(\dfrac{2}{3})=\dfrac{100}{27} \qquad =3{,}7 \Rightarrow (\dfrac{2}{3};3{,}7)$ $\qquad$ $[Substt\ \text{int}\, o\ original\, eqn]$

$\therefore\ f(3)=2(3)^3-11(3)^2+12(3)=-9 \quad \Rightarrow (3;-9)$

Local (or relative) Maximum and local minimum

Second derivative test can be used to determine the local maximum and local minimum points of the function:

1. Calculate $f'(x)$ and $f''(x)$;
2. Find all the stationery points of $f(x)$ by evaluating $f'(x)=0$;
3. Evaluate $f''(c)$ for each such stationery point c, and
   - If $f''(x) < 0$, then $f(x)$ has a local maximum at c
   - $f''(x) > 0$, then $f(x)$ has a local minimum at c.
   - $f''(x) = 0$, further investigation is necessary to determine what the nature of stationery point is.

**Example:**

Determine the local maximum and local minimum of the function $f(x)=x^3-3x^2-24x+32$ using the second derivative test.

Solution:

$f'(x)=3x^2-6x-24$

$f''(x)=6x-6$

Stationery points are found where $f'(x)=0$**. Then**

$3x^2-6x-24=0$

$3(x+2)(x-4)=0$

$x=-2 \quad and \quad x=4$

Now evaluate $f''(x)$ at the stationery points. Then

$$f''(-2) = 6(-2)-6 = -18 < 0 \Rightarrow local\max imum$$
$$f''(4) = 6(4)-6 = 18 > 0 \Rightarrow local\ \min imum$$

The corresponding f(x) values at the stationery points are:

$$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32 = 60$$
$$f(4) = -48$$

We can conclude that the point (−2; 60) is a local maximum and the point (4; −48) is a local maximum of *f*(*x*).

**ACTIVITY:**

Determine the local maximum and local minimum of the function $f(x) = 2x^3 - \frac{1}{2}x^2 - 12x - 10$ using the second derivative test.

SOLUTION:

$$f'(x) = 6x^2 - x - 12$$
$$f''(x) = 12x - 1$$
$$0 = 6x^2 - x - 12 \qquad [Stationery\ po\text{int}\, s]$$
$$(3x+4)(2x-3) = 0$$
$$x = -\frac{4}{3} \qquad and \qquad x = \frac{3}{2}$$
$$x = -1{,}333 \qquad and \qquad x = 1{,}5$$

Now for $x = -1,333$ we have:

$$f''(-1{,}333) = 12(-1{,}333) - 1 = -17 < 0 \qquad \Rightarrow local\ \max imum$$

The corresponding value of *f*(*x*) at $x = -1,333$ is

$$f(-1{,}333) = 2(-1{,}333)^3 - 0{,}5(-1{,}333)^2 - 12(-1{,}333) - 10 = 0{,}37$$

Therefore, the point $(-1{,}333; 0{,}37)$ is local maximum

Now for $x = 1,5$ we have:

$$f''(1{,}5) = 12(1{,}5) - 1 = 17 > 0 \qquad \Rightarrow local\ \min imum$$

The corresponding value of *f*(*x*) at $x = 1{,}5$ is

$$f(1{,}5) = 2(1{,}5)^3 - 0{,}5(1{,}5)^2 - 12(1{,}5) - 10 = -22{,}375$$

Therefore, the point $(1{,}5; -22{,}375)$ is local minimum

SUMMARY:

The following table shows the general characteristics of the function f(x) for various possible combinations of the signs of $f'(x)$ and $f''(x)$ on the interval (a; b):

Signs of $f'(x)$ and $f''(x)$	**Properties of the graph of *f*(*x*)**
$f'(x) > 0$ $f''(x) > 0$	*f*(*x*) increasing *f*(*x*) convex
$f'(x) > 0$ $f''(x) < 0$	*f*(*x*) increasing *f*(*x*) concave
$f'(x) < 0$ $f''(x) > 0$	*f*(*x*) decreasing *f*(*x*) convex
$f'(x) < 0$ $f''(x) < 0$	*f*(*x*) decreasing *f*(*x*) concave

C. INCREASING AND DECREASING FUNCTIONS

We use the terms ***increasing*** and ***decreasing*** to describe the behaviour of this function as we move from left to right along its graph. The graph in fig 1.1 shows the fuel economy of car as a function *f*(*x*) of its speed. Observe that the fuel economy, *f*(*x*), of the car improves as the speed of the car, x, increases from 0 to 60, and then drops as the speed increases beyond 60.

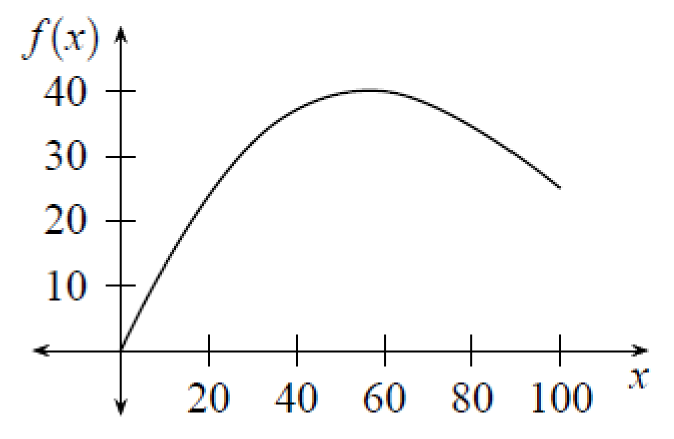


Figure 1.1 : Fuel economy of a car

A function *f*(*x*) is ***increasing*** on an interval (a; b) if, for any two numbers x_1 and x_2 in (a; b),

$f(x_1) < f(x_2)$ for $x_1 < x_2$. This is illustrated in Figure 1.2.

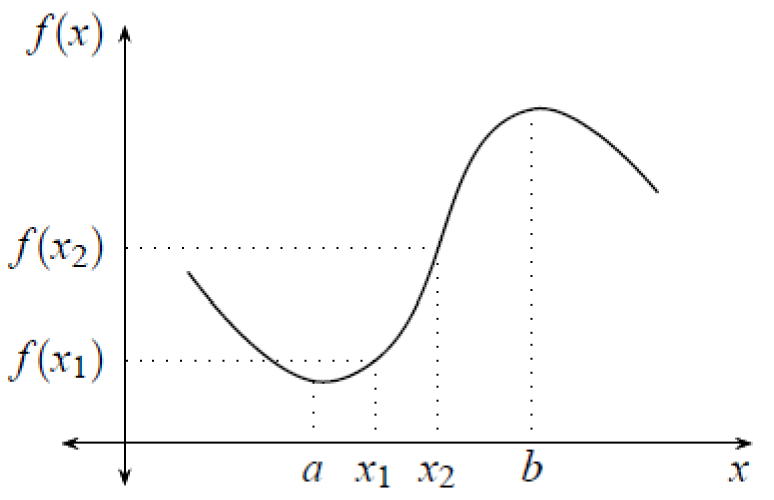


Figure 1.2 : Increasing function on $(a;b)$

A function $f(x)$ is ***decreasing*** on an interval $(a; b)$ if, for any two numbers x_1 and x_2 in (a; b), $f(x_1) > f(x_2)$ for $x_1 < x_2$. This is illustrated in Figure 1.3

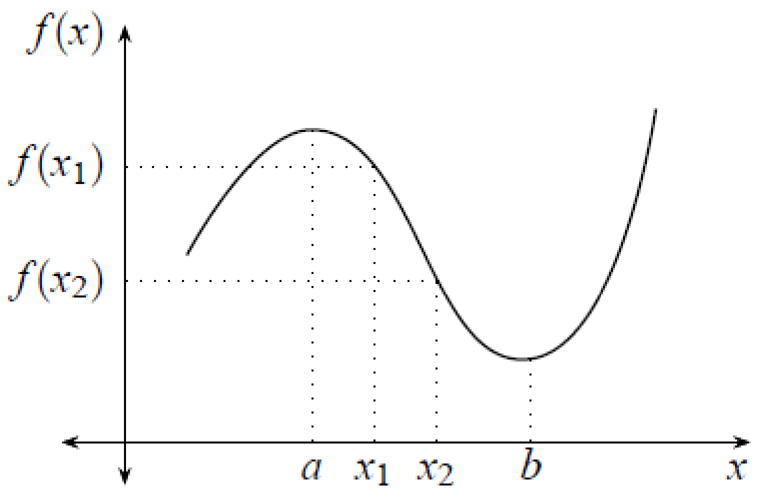


Figure 10.3: Decreasing function on $(a;b)$

The derivative can be used to determine the intervals where a differentiable function is ***increasing*** or ***decreasing***.

At a point where the ***derivative is positive***, the slope of the tangent line to the graph is positive and the ***function is increasing***.

At a point where the ***derivative is negative***, the slope of the tangent line to the graph is positive and the ***function is decreasing***.

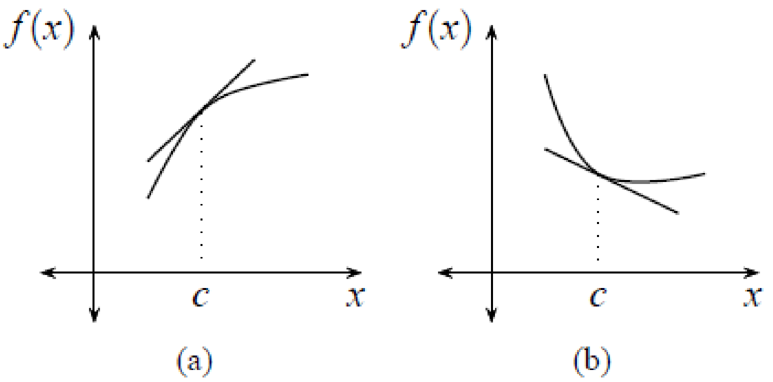


Figure 1.4 : Increasing/decreasing function and the slope of the tangent line

SUMMARY:

A. If $f'(x) > 0$ for each value of x in interval (a; b), then $f(x)$ is ***increasing*** on $(a; b)$;

B. If $f'(x) < 0$ for each value of x in interval (a; b), then $f(x)$ is ***decreasing*** on $(a; b)$;

C. If $f'(x) = 0$ for each value of x in interval (a; b), then $f(x)$ is ***constant*** on $(a; b)$

Examples:

1. Determine the interval where the function $f(x) = x^2$ is increasing and the interval where it is decreasing.

Solution:

$f'(x) = 2x$

Since $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, we can conclude that $f(x)$ is increasing on the interval $(0;\infty)$ and decreasing on the interval $(-\infty;0)$. This is confirmed by the graph in Figure 1.5

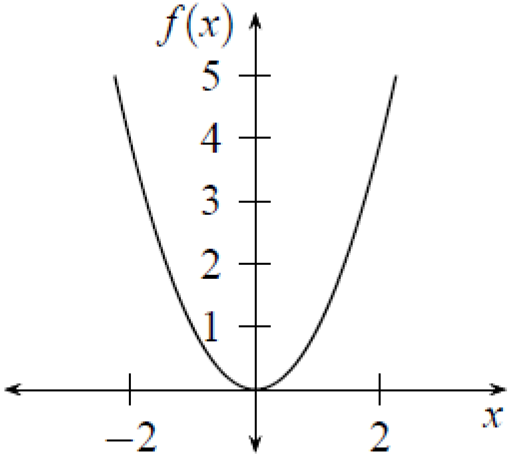


Figure 1.5 : Graph of $f(x) = x^2$

2. Determine the intervals where the function $f(x) = x^3 - 3x^2 - 24x + 32$ is increasing and where it is decreasing

Solution:

$$f'(x) = 3x^2 - 6x - 24$$
$$= 3(x+2)(x-4)$$
$$\therefore \quad x = -2 \quad or \quad x = 4$$

These points divide the real line into intervals $(-\infty; -2)$; $(-2; 4)$ and $(4; \infty)$

To determine the signs of $f'(x)$ in these intervals, we calculate the value of $f'(x)$ at a convenient test point in each interval. The results are shown below:

Interval	**Test point** c	$f'(x)$	**Sign of** $f'(x)$
$(-\infty; -2)$	**−3**	**21**	+
$(-2; 4)$	**0**	**−24**	**−**
$(4; \infty)$	**5**	**21**	+

Using these results, we can conclude that $f(x)$ is increasing on the intervals $(-\infty; -2)$ and $(4; \infty)$ and decreasing on the intervals $(-2; 4)$. This is confirmed by the graph on Figure 1.6

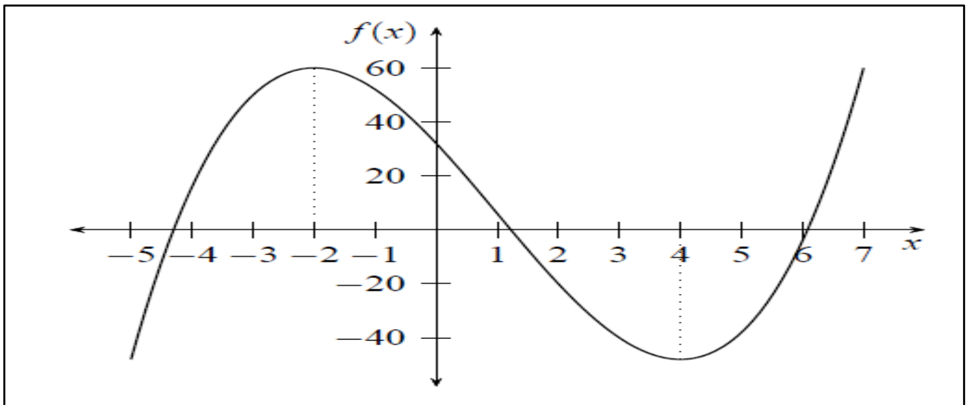


Figure 1.6 : Graph of $f(x) = x^3 - 3x^2 - 24x + 32$

**ACTIVITY**

Find the intervals where the following function is increasing and where it is decreasing:

$$f(x) = \frac{2}{3}x^3 - x^2 - 12x + 3$$

SOLUTION:

$$f'(x) = 2x^2 - 2x - 12$$
$$0 = 2x^2 - 2x - 12$$
$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$\therefore x = -2 \qquad and \qquad x = 3$$

These points divide the real line into the intervals $(-\infty; -2)$, $(-2;3)$ and $(3;\infty)$.

Determine the signs of $f'(x)$ in these intervals by calculating $f'(x)$ at a convenient test point in each interval. The results are as follows:

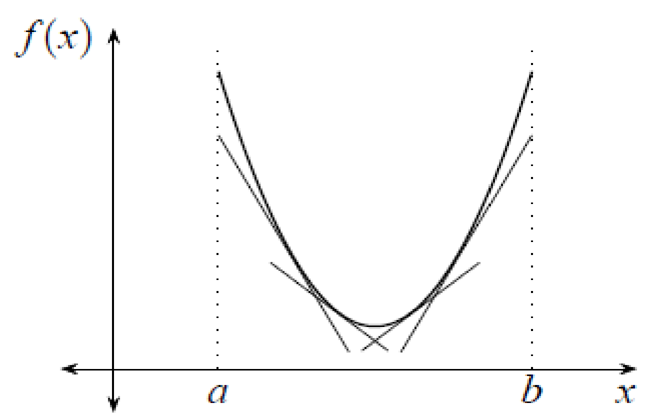
Interval	**Test point** c	$f'(x)$	**Signs of** $f'(x)$
$(-\infty; -2)$	**−3**	**12**	+
$(-2;3)$	**0**	**−12**	**−**
$(3;\infty)$	**4**	**12**	+

Therefore, f(x) is increasing over $(-\infty; -2)$ and $(3;\infty)$, and it is decreasing over $(-2;3)$

D. CONCAVITY

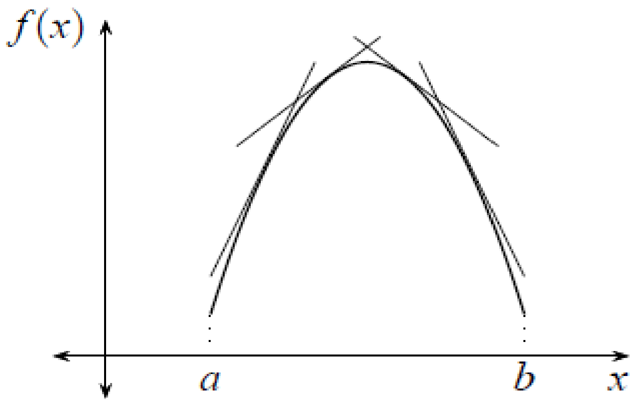
Concavity

The shape of the graph can be described using the notion of concavity. If a function is differentiable on an interval $(a; b)$, then $f(x)$ is ***concave upward*** (also called ***convex***) on $(a; b)$ if $f'(x)$ is ***increasing*** on $(a; b)$. A curve is concave upward (convex) if it lies above its tangent lines.



Graph of a convex function $f(x)$

If a function is differentiable on an interval $(a; b)$, then $f(x)$ is ***concave downward*** (also called ***concave***) on $(a; b)$ if $f'(x)$ is ***decreasing*** on $(a; b)$. A curve is concave downward (concave) if it lies below its tangent lines.



Graph of a concave function $f(x)$

We can use the second derivative $f''(x)$ to determine the concavity of a function $f(x)$.

NOTE: $f''(x)$ measures the rate of change of $f'(x)$ at the point x.

1. **If** $f''(x) > 0$ for each value of *x* in the interval (*a*; *b*), then *f*(*x*) is concave upward (**Convex)** on (a; b);
2. **If** $f''(x) < 0$ for each value of *x* in the interval (*a*; *b*), then *f*(*x*) is concave downward (**Concave)** on (*a*; *b*);

Example:

Determine the intervals where the function $f(x) = x^3 - 3x^2 - 24x + 32$ is convex and where it is concave.

Solution:

$$f'(x) = 3x^2 - 6x - 24$$
$$f''(x) = 6x - 6$$

If we set $f''(x) = 0$, then $x = 1$

This point divides the real line into two intervals $(-\infty;1)$ and $(1;\infty)$

We determine the signs of $f''(x) < 0$ in these intervals by calculating $f''(x)$ at a convenient test point in each interval. For example,

Select x = 0 in the interval $(-\infty;1)$. Then

$$f''(0) = -6 < 0$$
$$\Rightarrow f(x) \ is \ \ concave$$

Select x = 2 in the interval $(-\infty;1)$. Then

$$f''(2) = 6(2) - 6 = 6 > 0$$
$$\Rightarrow f(x) \ is \ \ convex$$

We can conclude that the function is concave over the interval $(-\infty;1)$ and convex over the interval $(1;\infty)$. Refer to Figure 1.6 above.

ACTIVITY

Determine where the following function is concave and where it is convex: $f(x) = 4x^3 - 3x^2 + 6$

SOLUTION:

$$f'(x) = 12x^2 - 6x$$
$$f''(x) = 24x - 6$$
$$0 = 24x - 6$$
$$\therefore\ x = \frac{1}{4}$$

This point divides the real line into two intervals $(-\infty; \frac{1}{4})$ and $(\frac{1}{4}; \infty)$.

Select *x* = 0 for the interval $(-\infty; \frac{1}{4})$. Then:

$$f''(0) = 24(0) - 6 = -6 < 0 \qquad \Rightarrow f(x)\ is\ concave$$

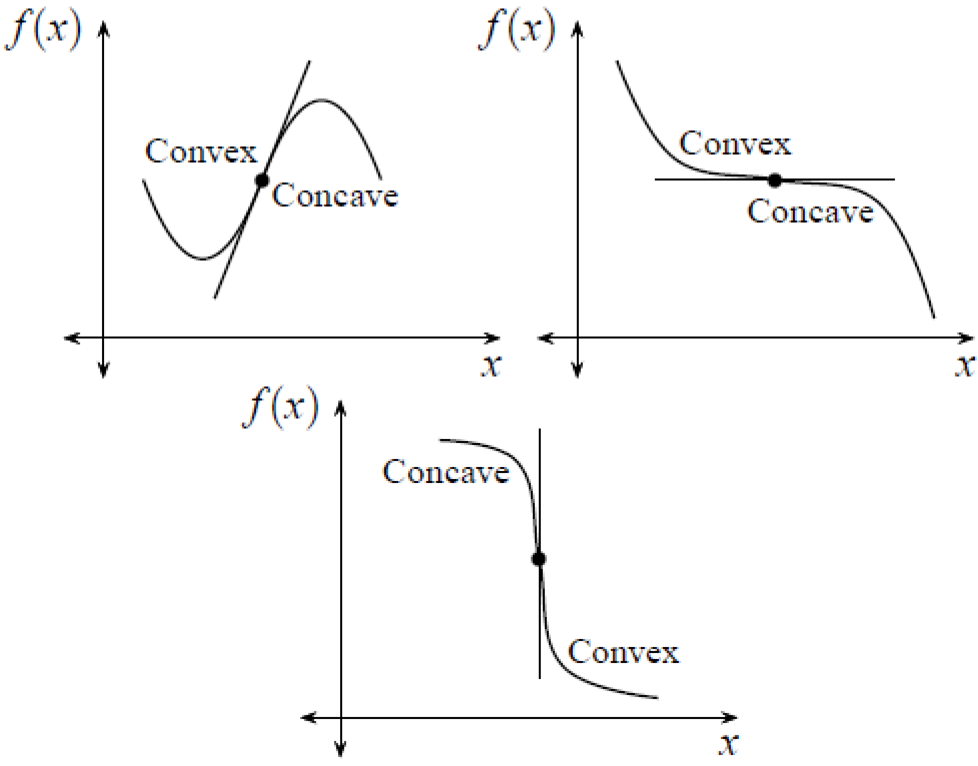
Select *x* = 1 for the interval $(\frac{1}{4}; \infty)$. Then:

$$f''(1) = 24(1) - 6 = 18 > 0 \qquad \Rightarrow f(x)\ is\ convex$$

E. POINTS OF INFLECTION

From the example above, it is clear that at the point x = 1 the function changes from being concave to convex. This point is called an ***inflection point*** of the function.

Def: The point on the graph of a differentiable function *f*(*x*) at which the concavity changes, is called an ***inflection point.*** If the cubic function has only one stationery point, this point will be a point of inflection that is also a stationery point. For points of inflection that are not stationery points, find $f''(x)$, equate it to 0 and solve for x. alternatively, simply add up the x-coordinates of the turning points and divide by 2 to get the x-coordinate of the point of inflection.



Inflection points

F. ZEROS OR ROOTS OF FUNCTIONS

Assume $f(x)$ is a continuous function. Any real number $\boldsymbol{r}$ for which $f(r) = 0$, is called the roots of the equation $f(x) = 0$ or the zero of the function f.

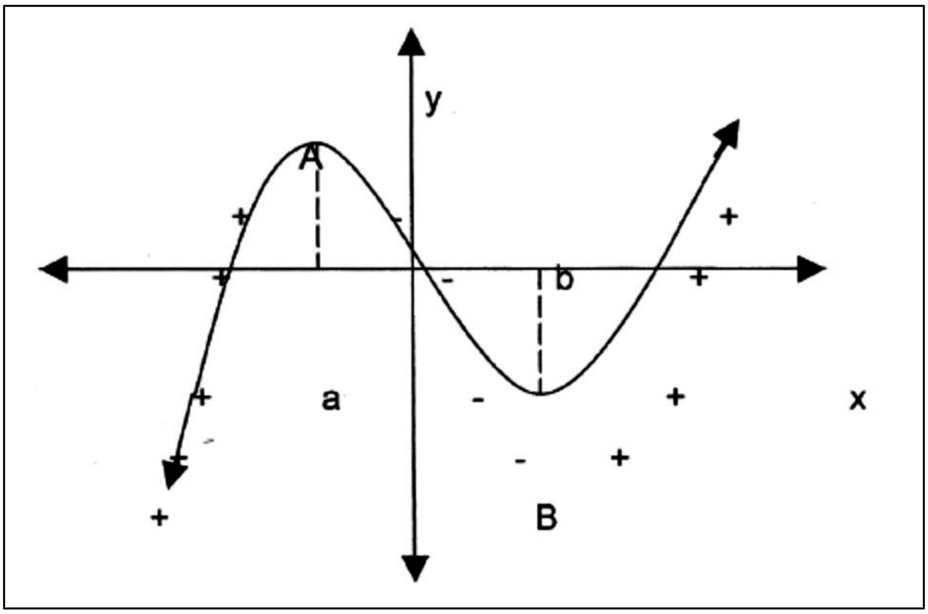
The zero of the quadratic function $f(x) = ax^2 + bx + c$ or the roots of the equation $ax^2 + bx + c = 0$ can be easily calculated with the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, to determine the solution (s) of polynomials of degree 3, we can calculate the roots using remainder and factor theorems and sometimes trial-and-improvement methods. For polynomials of degree 3 and higher, we mostly need computer based methods based on iterative procedures **(not relevant for Gr 12 level)**.

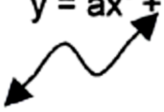
G. SKETCHING OF GRAPHS

General characteristics of a cubic graph



- At point A, the gradient is equal to zero [i.e., f'(x) = 0] and the point is local maximum;
- At point B, the gradient is equal to zero [i.e., f'(x) = 0] and the point is local minimum;
- For x < a, the gradient is positive [i.e., f'(x) > 0]
- For a < x < b, the gradient is negative [i.e., f'(x) < 0]
- For x > b, the gradient is positive [i.e., f'(x) > 0];

Shape: $y = ax^3 + bx^2 + cx + d$



If a > 0: and if a < 0:

General procedures for sketching graphs of functions *f*(*x*):

1. Determine the y-intercept: set $x = 0$
2. Determine the x-intercept: set $y = 0$.
3. Factorize equation and find values of x
4. If factors are not readily noticeable, use factor theorem to find factors;
5. Determine the turning points by using differentiation. At the turning point, set $f'(x) = 0$ and solve.
   Substitute the values of x from the above equation in the **ORIGINAL** equation to find the corresponding values of y.
6. Determine the intervals where f(x) is increasing and where it is decreasing;
7. Find the local maxima and minima of f(x);
8. Determine the concavity of f(x) and find the inflection point of f(x)
9. Plot additional points to help further identify the shape of the graph of f(x), and sketch the graph.

Example:

Given: $f(x) = -x^3 + 5x^2 + 8x - 12$

a. Use the factor theorem to show that x + 2 is a factor of *f*. hence solve $f(x) = 0$
b. Determine the stationery points for the graph of *f*.
c. Draw a neat sketch graph of *f*. clearly show intercepts and stationery points.
d. Determine the equation of the tangent to the graph of *f* at $x = -2$

Solution:

a.
$$\begin{aligned} f(-2) &= -(-2)^3 + 5(-2)^2 + 8(-2) - 12 \\ &= 0 \\ \Rightarrow \quad & x+2 \ is \ a \ \ factor \end{aligned}$$

$$(x+2)(-x^2 + 7x - 6) = 0$$
$$(x+2)(x-1)(x-6) = 0$$
$$\therefore \ x = -2 \ or \ x = 1 \ or \ \ x = 6$$

$$f(x) = -x^3 + 5x^2 + 8x - 12$$
$$f'(x) = -3x^2 + 10x + 8$$
$$-3x^2 + 10x + 8 = 0$$

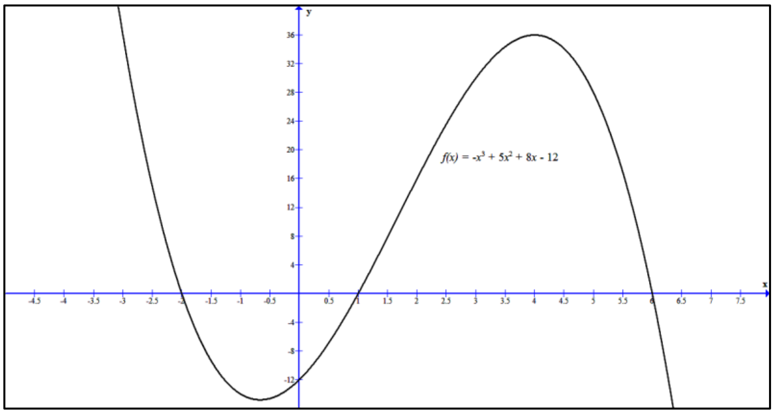
b. $$x = \left(\frac{-10 \pm \sqrt{(10)^2 - 4(-3)(8)}}{2(-3)} \right)$$

$$= \frac{-10 \pm \sqrt{196}}{6}$$

$x = -0.7 \qquad or \qquad x = 4$

$\therefore Stationery\ points: (-\frac{2}{3}; -\frac{400}{27}\ and\ (4; 36)$

c.



$$f(x) = -x^3 + 5x^2 + 8x - 12$$
$$f(-2) = -(-2)^3 + 5(-2)^2 + 8(-2) - 12 = 0$$
$$f'(-2) = -3(-2)^2 + 10(-2) + 8 = -24$$

d. $Equation\ of\ \tan gent: y = mx + c$

$0 = -24(-2) + c$

$\therefore c = 48$

$\therefore y = -24x + 48$

Teaching and learning tips (CPK/TPACK)

Concepts involved	***Important points to remember***	***Typical questions***	***Summary of skills and knowledge (Solution)***
Sketch graphs of cubic polynomial functions using differentiation to determine the co-ordinate of stationery points, and points of inflection (where concavity changes). Also, determine the x-intercepts of the graph using factor theorem and other techniques	Introduce the second derivative $f''(x)=\frac{d}{dx}(f'(x))$ of *f*(*x*) and how it determines the concavity of a function. To understand points of inflection, an understanding of concavity is necessary. This is where the second derivative plays a role.	Sketch the graph defined as $y=-x^3+4x^2-x$ by: (a) Finding the intercepts with the axes; (R) (b) Finding maxima, minima and the co-ordinate of the point of inflection; (C) (c) Looking at the behaviour of y as $x\to\infty$ and as $x\to-\infty$;(C)	• Find *x* and *y* intercepts of cubic curves, make *y* = 0 and solve the equation; • Find 1st factor using factor theorem; • Factorize further, may need quadratic formula; • Find local maxima and local minima turning points using $f'(x)=0$; • Find a point of inflection by making the 2nd derivative = 0 and substitute the x-value back into the original equation,

Summary Big Ideas

- Factorization
- Increasing and decreasing functions
- Stationery points
- Concavity
- Points of inflection
- Graph sketching

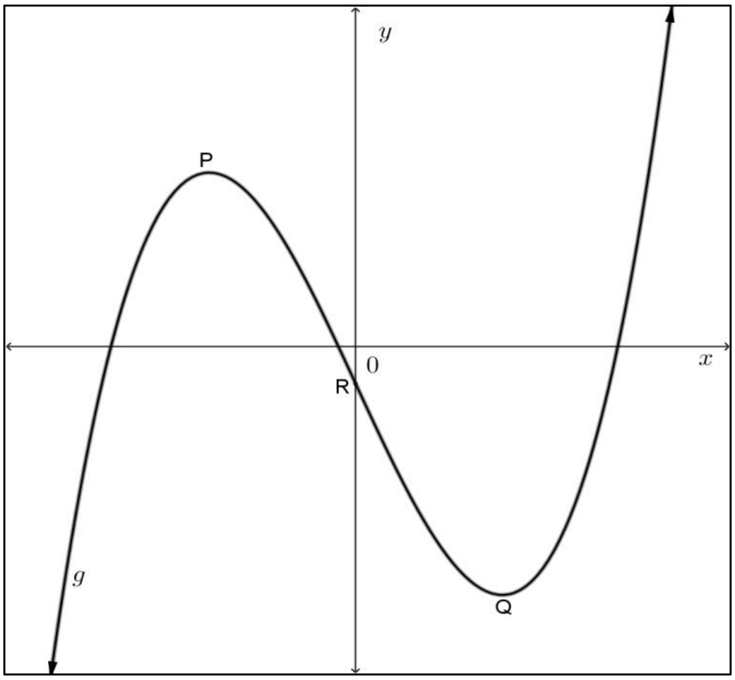
ICT Integration

Scientific calculator, www.graph.exe,

H. Resources: TYPICAL EXAM QUESTIONS

ACTIVITY 1

The sketch below shows the graph of $g(x) = x^3 - 6x - 1$. P and Q are the turning points and R the *y*-intercept of *g*.



1.1 Determine the coordinates of R. (2)

1.2 Determine the coordinates of the turning points P and Q. (6)

1.3 Calculate the values of x for which g strictly increases as x increases. (2)

1.4 If $h(x) = g'(x)$, determine for which values of x is $h(x) \leq 0$. (2)

1.5 Determine the equation of the tangent to *g* at R. (4)

1.6 Write down the equation of the line perpendicular to the tangent at P. (2)

[18]

ACTIVITY 2

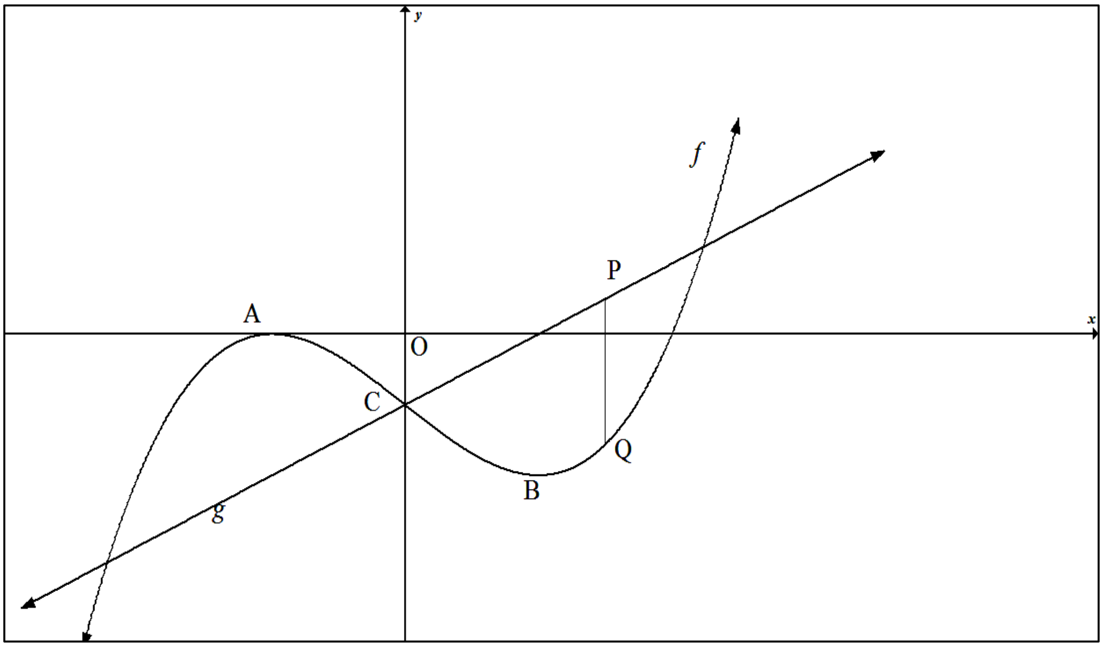
2.1 Given $f(x) = x^2 - 8$

2.1.1 Calculate $f(-3)$. (1)

2.1.2 Calculate $f^{/}(-3)$. (1)

2.1.3 Determine the equation of the tangent to $f(x) = x^2 - 8$ at $x = -3$. (2)

2.2 The graph of a cubic function with equation $f(x) = x^3 - 3x - 2$ and $g(x) = 2x - 2$ is drawn. A and B are the turning points of f. P is a point on g and Q is a point on f such that PQ is perpendicular to the x – axis.



2.2.1 Calculate the coordinates of A and B. (4)

2.2.2 If PQ is perpendicular to the x – axis, calculate the maximum length of PQ, (4)

2.2.3 Determine the values of k for which $f(x) = k$ has only two real roots. (2)

2.2.4 Determine the values of x for which f is concave up. (3)

[17]

ACTIVITY 3

Given: $f(x) = ax^3 + bx^2 + 3x + 3$ and $g(x) = f''(x)$ where $g(x) = 12x + 4$.

3.1 Show that $a = 2$ and $b = 2$. (4)

3.2 Prove that f will never decrease for any real value of x. (5)

3.3 Determine the minimum gradient of f. (4)

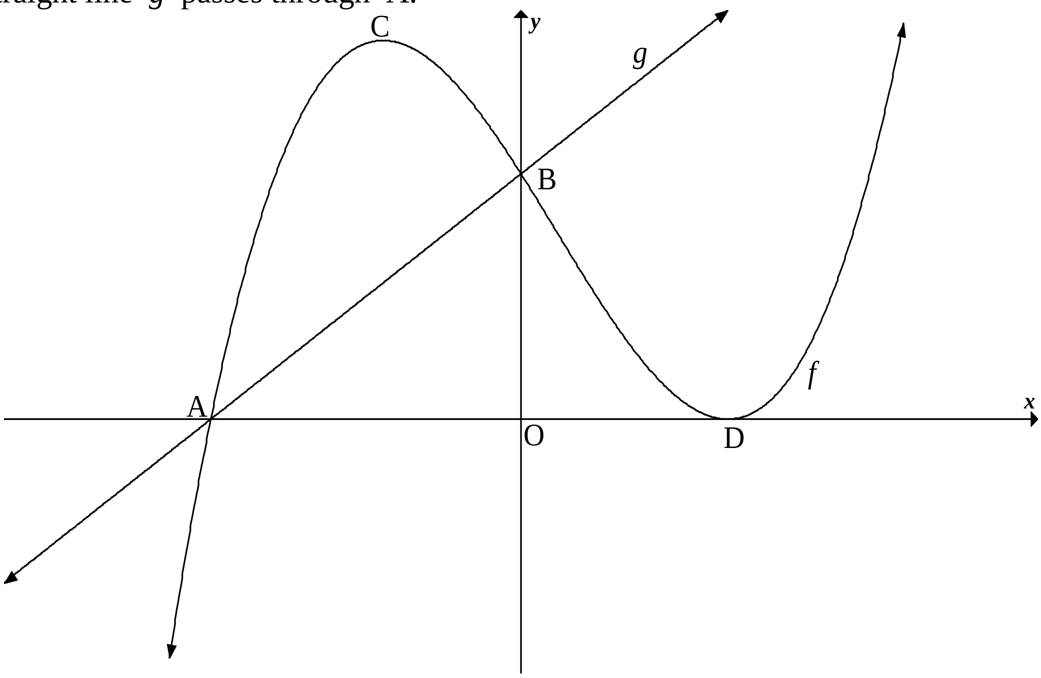
3.4 Explain the concavity of f for all values of x where $g(x) < 0$. (1)

[14]

ACTIVITY 4

Sketched below are the graphs of $f(x) = (x-2)^2(x-k)$ and $g(x) = mx + 12$

- A and D are the x-intercepts of f.
- B is the common y-intercept of f and g.
- C and D are turning points of f.
- The straight line g passes through A.



4.1 Write down the y-coordinate of B. (1)

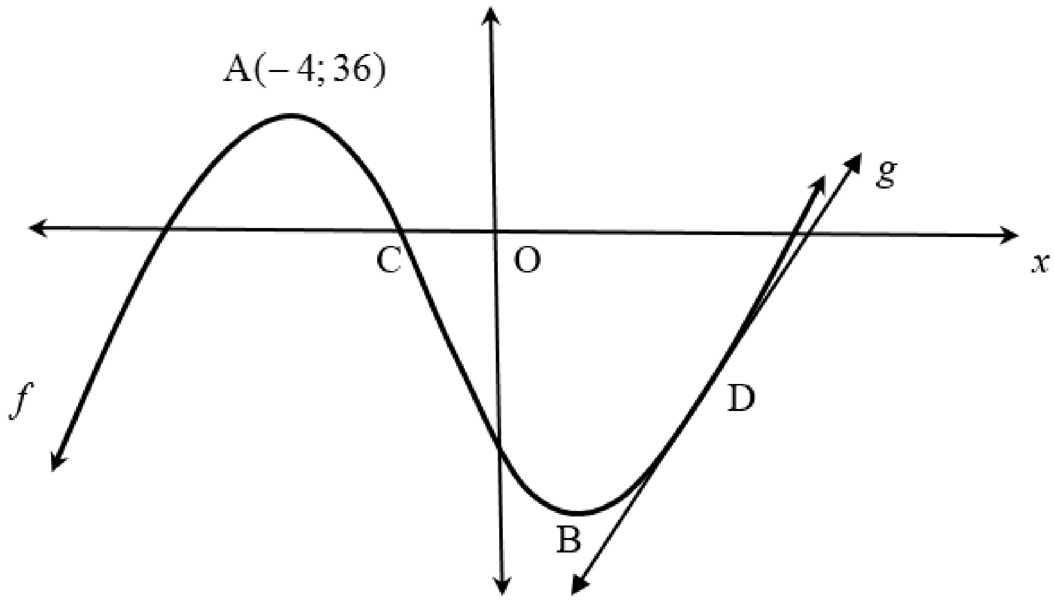
4.2 Calculate the x-coordinate of A. (3)

4.3 If $k = -3$, calculate the coordinates of C. (6)

4.4 For which values of x will f be concave down? (3)

[13]

ACTIVITY 5



5.1 Show that $p = 5$ and $q = -8$. (6)

5.2 If C(–1; 0) is an *x*-intercept of *f*, calculate the other *x*-intercepts of *f*. (4)

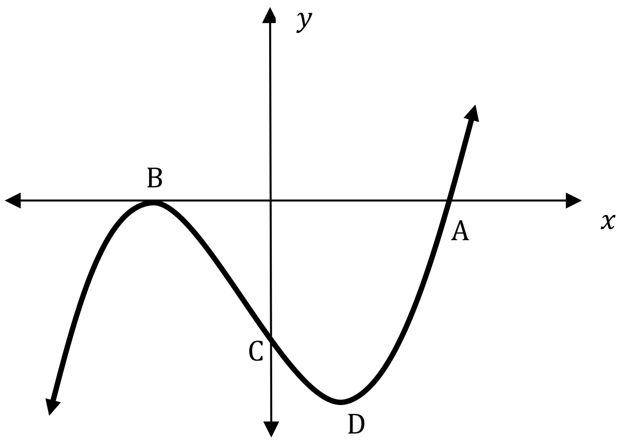
5.3 Determine the equation of *g*, the tangent to *f* at D(1 ; – 14). (4)

5.4 For which values of k will $f(x) = g(x) + k$ have two positive roots? (2)

[16]

ACTIVITY 6

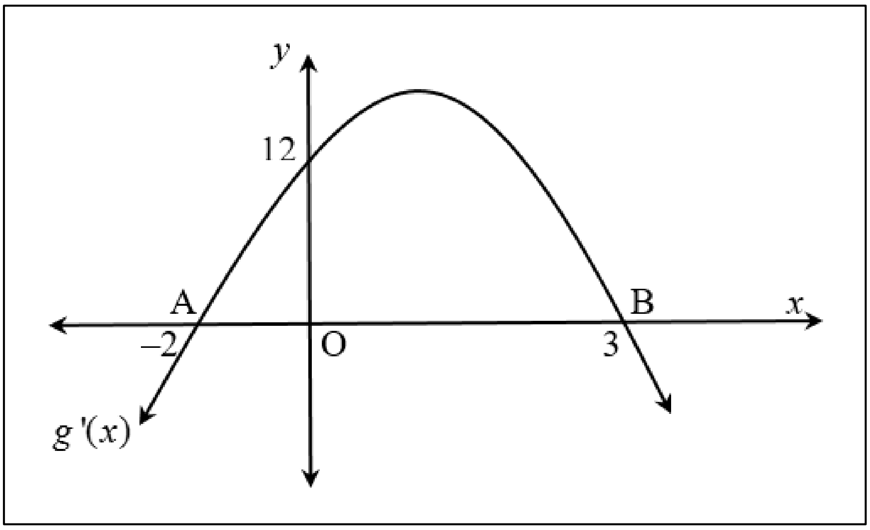
The diagram below shows the graph of $f(x) = x^3 + x^2 - x - 1$.



6.1	Calculate the distance between A and B, the x-intercepts.	(5)
6.2	Calculate the coordinates of D, a turning point of f.	(3)
6.3	Show that the concavity of f changes at $x = -\frac{1}{3}$	(3)
6.4	For which values of x is:	
	6.4.1 $f(x) > 0$	(1)
	6.4.2 $f(x).f'(x) < 0$	(3)
		[15]

ACTIVITY 7

The graph of $y = g'(x)$ is sketched below, with x-intercepts at A$(-2; 0)$ and B$(3; 0)$. The y-intercept of the sketched graph is $(0; 12)$.



7.1 Determine the gradient of g at $x = 0$. (1)

7.2 For which value of x will the gradient of g be the same as the gradient in QUESTION 7.2.1? (1)

7.3 Draw a sketch graph of $y = g(x)$. Show the x-values of the stationary points and the point of inflection on your sketch. It is not necessary to indicate the intercepts with the axes. (3)

[5]

SOLUTION TO ACTIVITIES

ACTIVITY 1

1.1	R $(0;-1)$	✓ $x=0$ ✓ $y=-1$ (2)
1.2	$g(x)=x^3-6x-1$ $g'(x)=3x^2-6$ $3x^2-6=0$ $x=\pm\sqrt{2}=\pm1{,}41$ $g(\pm\sqrt{2})=$ $y=\left(-\sqrt{2}\right)^3-6\left(-\sqrt{2}\right)-1$ or $y=\left(\sqrt{2}\right)^3-6\sqrt{2}-1$ $=-2\sqrt{2}+6\sqrt{2}-1$ or $=-1-4\sqrt{2}$ $=4{,}66$ $=-6{,}66$ P $(-1{,}41;\,4{,}66)$ Q $(1{,}41;-6{,}66)$	✓ $3x^2-6$ ✓ $3x^2-6=0$ ✓ $x=\pm1{,}41$ ✓substitution ✓ P $(-1{,}41;\,4{,}66)$ ✓ Q $(1{,}41;-6{,}66)$ (6)
1.3	$x<-\sqrt{2}$ or $x>\sqrt{2}$ **OR** $x\in(-\infty;-1{,}41)$ or $x\in(1{,}41;\infty)$	✓ $x<-\sqrt{2}$ ✓ $x>\sqrt{2}$ (2)
	$-\sqrt{2}\le x\le\sqrt{2}$	✓critical values ✓notation (2)
1.5	$g'(x)=3x^2-6$ and $x=0$ at R $m_t=g'(0)=3(0)^2-6=-6$ $y-y_1=-6(x-x_1)$ $y+1=-6(x-0)$ $y=-6x-1$	✓ $g'(x)$ ✓ $m_t=-6$ ✓subst. $(0;-1)$ ✓ $y=-6x-1$ (4)
1.6	$h(x)=\frac{1}{6}x-1$	✓ $m=\frac{1}{6}$ ✓equation (2)
		[18]

ACTIVITY 2

2.1.1	$f(-3)=(-3)^2-8=1$	A✓ $y=1$ (y – value)	(1)
2.1.2	$f'(x)=2x$ $m=f'(-3)=2(-3)=-6$	A✓ value of gradient of tangent	(1)

2.1.3	Equation of tangent : - $y = mx + c$ $1 = -6(-3) + c$ $c = -17$ $y = -6x - 17$ **OR** $y - 1 = -6(x + 3)$ $y = -6x - 18 + 1$ $y = -6x - 17$	CA✓ substitution CA✓answer CA✓ substitution CA✓answer	(2) (2)
2.2.1	$f(x) = x^3 - 3x - 2$ $f'(x) = 3x^2 - 3 = 0$ $3(x+1)(x-1) = 0$ $x = -1 \quad or \quad x = 1$ $y = 0 \quad or \quad y = -4$ $A(-1\,;\,0) \qquad B(1\,;\,-4)$	A✓derivative <u>and</u> equating to 0 CA✓factors CA✓x - values CA✓ y - values	(4)
2.2.2	$PQ = g(x) - f(x)$ $= (2x - 2) - (x^3 - 3x - 2)$ $= -x^3 + 5x$ $PQ' = -3x^2 + 5 = 0$ $x^2 = \frac{5}{3}$ $x = \pm\sqrt{\frac{5}{3}} = 1{,}29$ $For \quad x = 1{,}29$ $y = -(1{,}29)^3 + 5(1{,}29) = 4{,}3$ $Maximum\,Length\,of\;\; PQ = 4{,}3 units$	A✓expression for PQ A✓ $PQ' = 0$ CA✓ x – values CA✓y value for x = 1,29	(4)
2.2.3	$k = -4 \quad or \quad k = 0$	$k = 0$ A✓ k = -4 CA✓answers	(2)
2.2.4	$f''(x) = 6x > 0$ $x > 0$ **OR** $x = \frac{x_1 + x_2}{2} = \frac{(-1)+(1)}{2} = 0$ $x > 0$	A✓ $6x$ A✓ $6x > 0$ A✓answer A✓midpoint formula	(3)

	OR $x=-\frac{b}{3a}=-\frac{0}{3(1)}=0$ $x>0$	CA✓ $x=0$ CA✓answer A✓formula A✓ $x=0$ A✓answer	(3) (3)
			[17]

ACTIVITY 3

3.1		$f(x)=ax^3+bx^2+3x+3$ $f'(x)\ =3ax^2+2bx+3$ $f''(x)=6ax+2b$ but $f''(x)=\ 12x\ +4.$ so $6a=12\ \Rightarrow a=\ 12$ $2b=4\ \Rightarrow b=2$	✓ ✓ ✓ ✓	$f'(x)$ $f''(x)=6ax+2b$ $6a=12$ $2b=4$	(4)
3.2		$f(x)=2x^3+2x^2+3x+3$ $f'(x)=6x^2+4x+3$ now $f'(x)=0\ if\ 6x^2+4x+3=0$ $ie\ if\ x=\frac{-4\pm\sqrt{(4)^2-4(6)(3)}}{2(6)}$ $x=\ -\frac{4\pm\sqrt{-56}}{12}$ x is imaginary/non- real graph has no turning points.	✓ ✓ ✓ ✓	$f'(x)=6x^2+4x+3$ $6x^2+4x+3=0$ subst. into formula graph has no turning points. but $a>0$, graph starts by increasing $a>0$,	(5)

		but $a > 0$, graph starts by increasing $a > 0$, OR $f(x) = 2x^3 + 2x^2 + 3x + 3$ $f'(x) = 6x^2 + 4x + 3$ now $f'(x) = 0$ if/as $6x^2 + 4x += 0$ $\Delta = b^2 - 4ac$ $= (4)^2 - 4(6)(3)$ $= -56$ roots are imaginary - graph has no turning points./ $a > 0$ graph starts by increasing/ $a > 0$, OR $f'(x) = 6x^2 + 4x + 3$ $= 6\left(x + \frac{1}{3}\right)^2 + \frac{7}{3}$ but $\left(x + \frac{1}{3}\right)^2 \geq 0$ so $6\left(x + \frac{1}{3}\right)^2 \geq 0$ and $6\left(x + \frac{1}{3}\right)^2 + \frac{7}{3} \geq 0$ $f'(x) > 0$ for all $x \in \Re$	✓ OR ✓ ✓ ✓ ✓ ✓ OR ✓ ✓ ✓ ✓ ✓	OR $f'(x) = 6x^2 + 4x + 3$ $6x^2 + 4x + 3 = 0$ $\Delta = (4)^2 - 4(6)(3)$ $= -56$ and graph has no turning points. $a > 0$, graph starts by increasing/ $a > 0$ OR $f'(x) = 6x^2 + 4x + 3$ completing the square reasoning $\left(x + \frac{1}{3}\right)^2 \geq 0$ reasoning $6\left(x + \frac{1}{3}\right)^2 + \frac{7}{3} \geq 0$ conclusion	
3.3		the gradient $f'(x) = 6x^2 + 4x + 3$			

		is minimum when $f''(x) = 0$/ $12x + 4 = 0$ $x = -\frac{1}{3}$ then $f'\left(-\frac{1}{3}\right) = 6\left(-\frac{1}{3}\right)^2 + 4\left(-\frac{1}{3}\right) + 3$ $= \frac{7}{3}$	✓ ✓ ✓ ✓	$12x + 4 = 0$ $x = -\frac{1}{3}$ substitution $= \frac{7}{3}$	(4)
3.4		$g(x) < 0 \implies f''(x) < 0$ $x < -\frac{1}{3}$ graph is concave down	✓	concave down	(1)
					[14]

ACTIVITY 4

4.1	$y = 12$	✓answer (1)
4.2	$12 = (0-2)^2(0-k)$ $k = -3$ $(x-2)^2(x+3) = 0$ $x = -3$ **OR/OF**	✓ substituting (0;12) ✓ $y = 0$ ✓ $x = -3$ (3) ✓ $y = 0$

	$y=0$ $(x-2)^2(x-k)=0$ $(x^2-4x+4)(x-k)=0$ $x^3-kx^2-4x^2+4kx+4x-4k=0$ But $-4k$ is the y-intercept $-4k=12$ $k=-3$ $x=-3$	✓ $-4k=12$ ✓ $x=-3$ (3)
4.3	$f(x)=x^3+3x^2-4x^2-12x+4x+12$ $f(x)=x^3-x^2-8x+12$ $f'(x)=3x^2-2x-8$ $3x^2-2x-8=0$ $(3x+4)(x-2)=0$ $x=-\frac{4}{3}$ or $x=2$ $y=\frac{500}{27}$ or $18{,}52$ or $18\frac{14}{27}$ $C\left(-\frac{4}{3};18{,}52\right)$	✓ $f(x)=x^3-x^2-8x+12$ ✓derivative ✓derivative equal to 0 ✓factors ✓x values ✓coordinates of C (6)
4.4	$f''(x)=6x-2$ $6x-2<0$ $x<\frac{1}{3}$ f is concave down when $x<\frac{1}{3}$ **OR/OF** $f''(x)=6x-2$ $6x-2=0$ $x=\frac{1}{3}$	✓second derivative ✓value of x ✓answer (3)

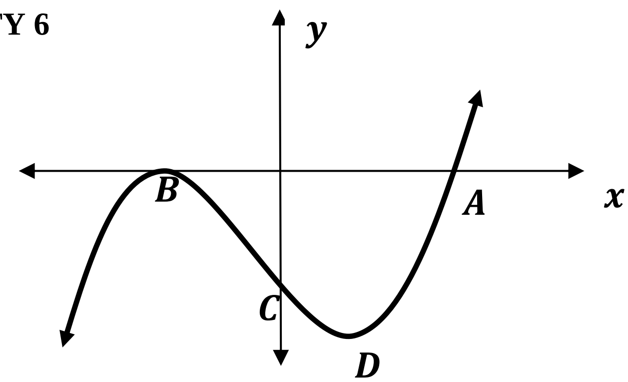
	f is concave down when $x < \frac{1}{3}$	✓second derivative ✓value of x ✓answer (3)
		[13]

ACTIVITY 5		
5.1.1	$f(x) = x^3 + px^2 + qx - 12$ $36 = (-4)^3 + p(-4)^2 + q(-4) - 12$ $36 = -64 + 16p - 4q - 12$ $112 = 16p - 4q$ $28 = 4p - q$ ....(1) $f^{/}(x) = 3x^2 + 2px + q$ $f^{/}(-4) = 3(-4)^2 + 2p(-4) + q$ $0 = 48 - 8p + q$ $-48 = -8p + q$ ....(2) $28 = 4p - q$ ....(1) $(1) + (2):$ $-20 = -4p$ $5 = p$ $(1): 28 = 4(5) - q$ $q = -8$	✓ substitution of $(-4; 36)$ in $f(x)$ ✓ $28 = 4p - q$ ✓ $f^{/}(x) = 3x^2 + 2px + q$ ✓ $f^{/}(-4) = 0$ ✓ $48 = 8p - q$ ✓ simplification

		(6)
5.2	$f(x) = x^3 + 5x^2 - 8x - 12$ $f(x) = (x+1)(x^2 + 4x - 12)$ (*long division method*) $0 = (x+1)(x^2 + 4x - 12)$ $0 = (x+1)(x+6)(x-2)$ $x = -1$ or $x = -6$ or $x = 2$ Coordinates of other x-intercepts are: $(-6;\ 0)$ or $(2;0)$ **OR** $(x+1)(x^2 + bx - 12) = x^3 + 5x^2 - 8x - 12$ $bx - 12x = -8x$ $b - 12 = -8$ $\therefore b = 4$ $x^2 + 4x - 12 = 0$ $(x+6)(x-2) = 0$ $x = -6$ or $x = 2$ Coordinates of other x-intercepts are: $(-6;\ 0)$ or $(2;0)$	✓ $(x+1)(x^2 + 4x - 12)$ ✓ factors ✓ $(-6;\ 0)$ ✓ $(2;0)$ (4) ✓ b = 4 $(x+1)(x^2 + 4x - 12)$ ✓ factors ✓ $(-6;\ 0)$ ✓ $(2;0)$ (4)

5.3	$f'(x) = 3x^2 + 10x - 8$ $f'(1) = 3(1)^2 + 10(1) - 8$ $= 5$ $y - y_1 = m(x - x_1)$ $y + 14 = 5(x - 1)$ $y = 5x - 19$ **OR** $f'(x) = 3x^2 + 10x - 8$ $f'(1) = 3(1)^2 + 10(1) - 8$ $= 5$ $y = mx + c$ $-14 = 5(1) + c$ $-19 = c$ $\therefore y = 5x - 19$	✓ $f'(x) = 3x^2 + 10x - 8$ ✓ $f'(1) = 5$ ✓ substitution ✓ answer (4) ✓ $f'(x) = 3x^2 + 10x - 8$ ✓ $f'(1) = 5$ ✓ substitution ✓ answer (4)
5.4	$0 < k \leq 7$	✓ $0 < k$ ✓ $k \leq 7$ (2) **[16]**

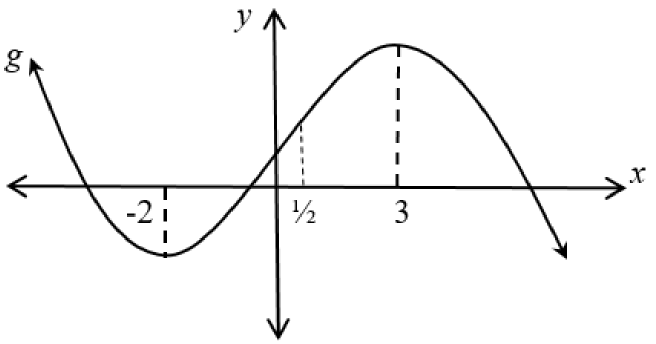
ACTIVITY 6



6.1	$f(x) = x^3 + x^2 - x - 1$ $\therefore (x-1)(x^2+2x+1) = 0$ $\therefore (x-1)(x+1)(x+1) = 0$ $\therefore x = 1$ or/*of* $x = -1$ $\therefore$ AB $= 2$ units	✓$(x-1)$ ✓(x^2+2x+1) ✓Factors = *0* ✓Both roots ✓$AB = 2$	(5)
6.2	$f'(x) = 0$ $\therefore 3x^2 + 2x - 1 = 0$ $\therefore (3x-1)(x+1) = 0$ $\therefore x = \frac{1}{3}$ or / *of* $x = -1$ $\therefore y = -\dfrac{32}{27}$ $\therefore D\left(\dfrac{1}{3}; -\dfrac{32}{27}\right)$	✓Derivative = 0 ✓ Values of x ✓Coordinates of D	(3)
6.3	$f''(x) = 0$ $\therefore 6x + 2 = 0$ $\therefore x = -\frac{1}{3}$ $h''(x) < 0$ | $h''(x) > 0$ $-\frac{1}{3}$ The concavity changes at $x = -\frac{1}{3}$	✓ Linking second derivative with concavity ✓ $6x + 2 = 0$ ✓ explanation	(3)
6.4.1	$x > 1$	✓ Answer	(1)
6.4.2	$x < -1$ or/*of* $-\frac{1}{3} < x < 1$	✓ $x < -1$ ✓✓ $-\frac{1}{3} < x < 1$ • Notation (-1) /	(3)
			[15]

ACTIVITY 7

7.1	$g'(0) = 12$	✓ answer (1)
7.2	symmetry at $x = \dfrac{1}{2}$ $\therefore x = 1$	✓ $x = 1$

		(1)
7.3		✓ turning point at $x = -2$ and at $x = 3$ ✓ point of inflection at $x = ½$ ✓ shape (3)

Where are we now?

Module 1	**Module 2**	**Module 3**	**Module 4**
Algebraic functions & graphs	*Cubic functions*	*Trigonometric Graphs*	*Financial Mathematics*

TRIGONOMETRIC GRAPHS

Overall aim of the module

The aims of teaching and learning mathematics are to encourage and enable teachers to:

- recognize that mathematics permeates the world around us
- appreciate the usefulness, power and beauty of mathematics
- enjoy mathematics and develop patience and persistence when solving problems
- understand and be able to use the language, symbols and notation of mathematics
- become confident in using mathematics to analyse and solve problems both in school and in real-life situations
- develop the knowledge, skills and attitudes necessary to pursue further studies in mathematics
- develop critical thinking and the ability to reflect critically upon their work and the work of others
- develop a critical appreciation of the use of information and communication technology in mathematics

Learning objectives of the module

At the end of the Module (**Trig graphs**) you will be able to:

After completing this topic teachers should be able to:

- Draw the trigonometric graphs:
- Make deductions from graphs (graphical interpretations);
- Identify amplitude, period, domain and range from the graphs;

Why study trigonometric graphs?

- ➢ The graphs are probably the most commonly used tools in all areas of science and engineering. They are used for **modelling** many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals, etc).
- ➢ The trigonometric graphs are **periodic**, which means the shape repeats itself exactly after a certain amount of time. Anything that has a **regular cycle** (like the tides, temperatures, rotation of the earth, etc) can be modelled using a sine or cosine curve

Key Content Addressed by this module: ATP (Show curriculum mapping)

Topic	Concepts	Prior Knowledge	Methodology Teaching Strategies	Vocabulary	Assessment & Resources
Trigonometric graphs	• Characteristics • Graph sketching • Graphical	Gr 10 Revision Function & mapping notations	Investigative approach Class activities Discussion methods		Examples Class activities Exam type questions

WEIGHTINGS OF CONTENT AREAS			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Differential Calculus			35 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3

TERMS	**GRADE 10**		**GRADE 11**		**GRADE 12**	
	Topic	**No of weeks**	**Topic**	**No of weeks**	**Topic**	**No of weeks**
TERM 1	Trigonometry	(3 Weeks)			Trigonometry	(2 Weeks)
TERM 2	Trig. Functions	(1 Week)	Trig Graphs and Equations included	(4 Weeks)	Trigonometry	(2 Weeks)
TERM 3	Trigonometry	(2 Weeks)	Sine, Cosine and Area Rules	(2 Week)	Functions (Polynomials) and Differential Calculus	(1 Week) (3 Weeks)

Activities/ Tasks: Trigonometric graphs

INVESTIGATION: TRIGONOMETRIC GRAPHS

Investigate how trigonometric graphs are affected by changing coefficients or constants in the given equations.

ACTIVITY 1:

Complete the table and draw the following graphs on the same set of axis:

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y = \sin x$													
$y = -2\sin x$													
$y = 3\sin x$													
$y = \frac{1}{2}\sin x$													
$y = \cos x$													
$y = -2\cos x$													
$y = 3\cos x$													
$y = \frac{\cos x}{2}$													

1.1. Investigate 'a' in the graphs:

$y = \sin x$

$y = -2\sin x$

$y = 3\sin x$

$y = \frac{1}{2}\sin x$

1.2. Investigate 'a' in the graphs:

$y = \cos x$

$y = -2\cos x$

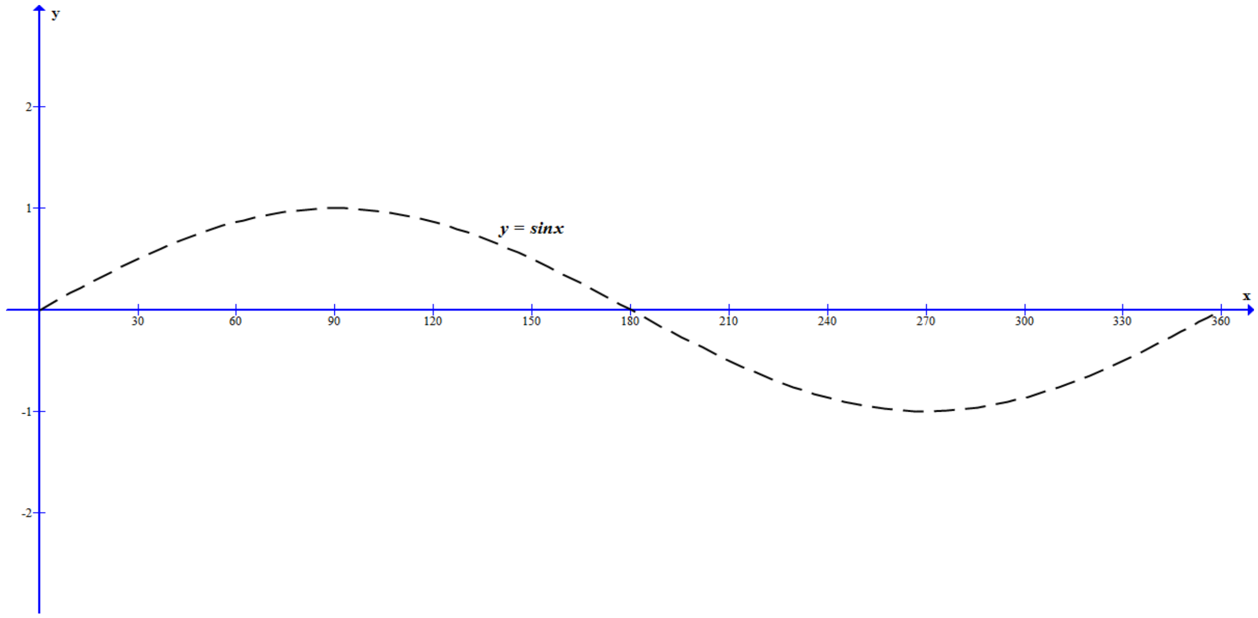
$y = 3\sin x$

$y = \frac{1}{2}\cos x$

GROUP DISCUSSIONS

BASIC TOOLS FOR TRIGONOMETRIC FUNCTIONS:

(a) $y = \sin x$ for the domain $x \in [0°; 360°]$



+ + — —	**Signs in 4 quadrants**	**+ means graph is above *x*- axis** **- means graph is above *x*- axis**

Shape: Wave-like shape, starting at the origin

Intercepts: y-intercept = 0

x-intercept **(zeros)** = $0°; 180°; 360°$ (every $180°$ starting at $0°$)

Domain: The domain is usually limited to the interval $[0°; 360°]$ or $[-360°; 360°]$

Infinite angles are possible as a line centred at the origin on the Cartesian plane can be rotated many times. Rotating the line anti-clockwise gives positive angles, and rotating clockwise gives negative angles.

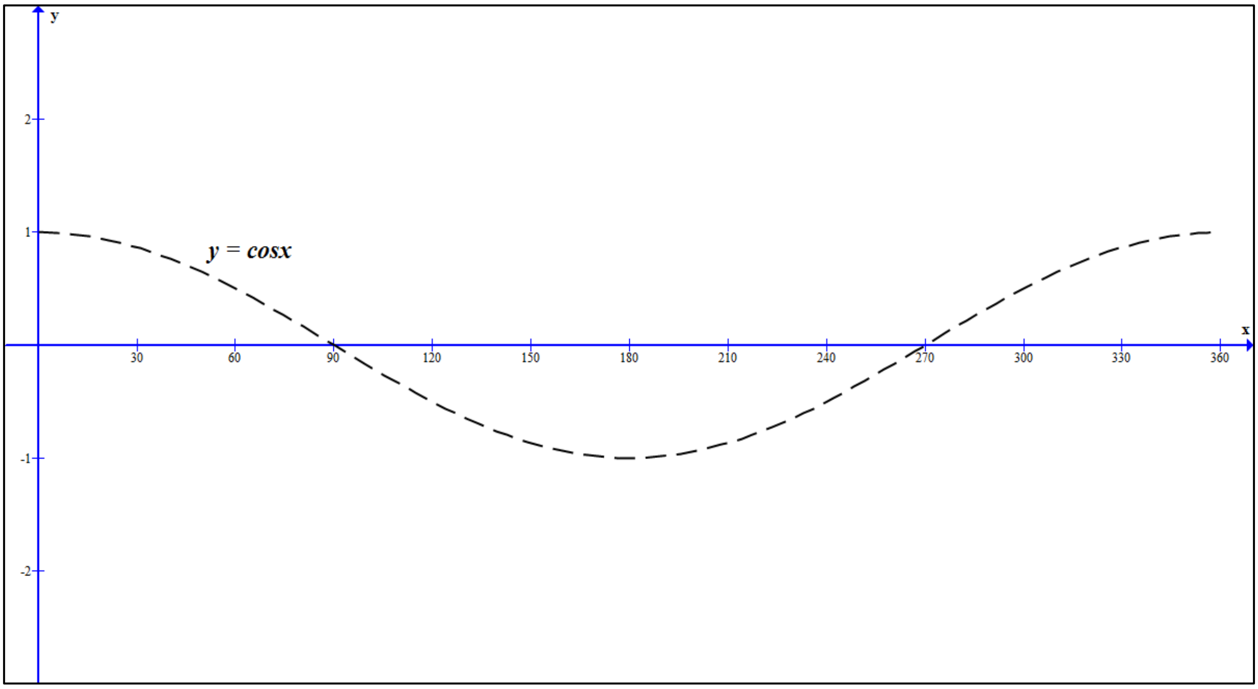
Period: The period of the function is the number of degrees the function needs to complete one cycle. This corresponds to one rotation. The sine function repeats itself every $360°$.

Trigonometric Functions are periodic functions and the graph form $0°$ to $360°$ is exactly the same as the graph from $360°$ to $720°$

Range: Minimum value: -1 when the angle x is $270°$ or $-270°$
Maximum value: 1 when the angle x is $90°$ or $-90°$
$[-1;\ 1]$

Amplitude: The amplitude of a trigonometric graph is the greatest distance the function moves above or below the x-axis.
(This only applies for sin or cos graphs). For the basic graph (also known as parent function) the amplitude is 1. It is half the range $(i.e., \frac{1+1}{2}=1)$.

(b) $y=\cos x$ for the domain $x\in[0°;360°]$



+ + / − − **Signs in 4 quadrants**	**+ means graph is above *x*- axis** **- means graph is above *x*- axis**

Shape: Wave-like shape, but when $x=0°$ the graph is at 1

Intercepts: y-intercept = 1
x-intercept **(zeros)** = $90°;270°$ (every $180°$ starting at $90°$)

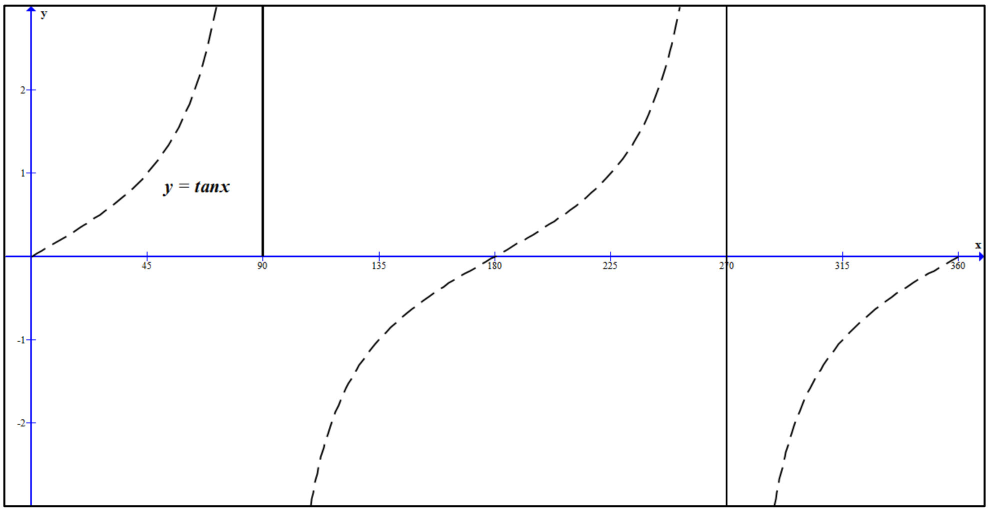
Domain: The domain is usually restricted to the interval $[0°;360°]$ or $[-360°;360°]$

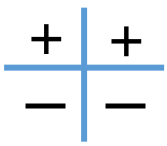
Period: This corresponds to one rotation. The cosine function repeats itself every $360°$

Range: Minimum value: -1 when the angle x is $180°$ or $-180°$
Maximum value: 1 when the angle x is $0°$, $360°$ and $-360°$
$[-1;\ 1]$

Amplitude: For the basic graph the amplitude is 1. It is half the range.

(c) $y = \tan x$ for the domain $x \in [0°; 360°]$





Signs in 4 quadrants

+ means graph is above *x*- axis

- means graph is above *x*- axis

Shape: **Not** wave-like shape, but long thin curve that is repeated

Intercepts: *y*-intercept = 0

x-intercept = $0°;180°$ (every $180°$ starting at $0°$)

Domain: The domain is usually restricted to the interval $[0°;360°]$ or $[-360°;360°]$

Asymptotes: at $x = \pm 90°$ and $x = \pm 270°$, the function is undefined. So, make sure that the graph does not touch or cross over the line $x = \pm 90°$ and $x = \pm 270°$

Period: The tangent function repeats itself every $180°$, starting at $-90°$ to $90°$

Range: $(-\infty; \infty)$

Minimum and Maximum values occur at the asymptotes at $x = \pm 90°$ and $x = \pm 270°$ (every $180°$ starting at $90°$). The maximum and minimum values of tan *x* are not important as tan *x* could be any real number.

Amplitude: Since the tangent function is not a wave-like graph, it does not have an amplitude. However, for the parent function when $x = 45°$, the value of the function is 1

Summary of the Basic characteristics of trigonometric graphs

Finding of	$y=a\sin bx$	$y=a\cos bx$	$y=a\tan bx$
Domain	$[-360°;360°]$	$[-360°;360°]$	$(-90°;0°],(-90°;-270°)$, $(-270°;360°],[0°;90°)$, $(90°;270°),(270°;360°]$
Range	$[-1;1]$	$[-1;1]$	$[-\infty;\infty]$
Period	$\frac{360°}{\lvert b\rvert}=\frac{360°}{\lvert 1\rvert}=360°$	$\frac{360°}{\lvert b\rvert}=\frac{360°}{\lvert 1\rvert}=360°$	$\frac{180°}{\lvert b\rvert}=\frac{180°}{\lvert 1\rvert}=180°$
Amplitude	$a=1$	$a=1$	No
Minimum value	-1	-1	No
Maximum value	1	1	No
Asymptote	NO	NO	$x=-90°, x=-270°$ $x=90°, x=270°$
NOTE: the value of *b* affects the period only and the amplitude (maximum & minimum) remains the same.			

Variations of graphs [Translations (moving the axes)]

1. Changing the amplitude [The effects of *a*]

$y=a\sin\theta$, $y=a\cos\theta$, $y=a\tan\theta$

Def: ***Amplitude*** is defined as $\frac{1}{2}$ (the distance between the maximum and the minimum points of the curve.

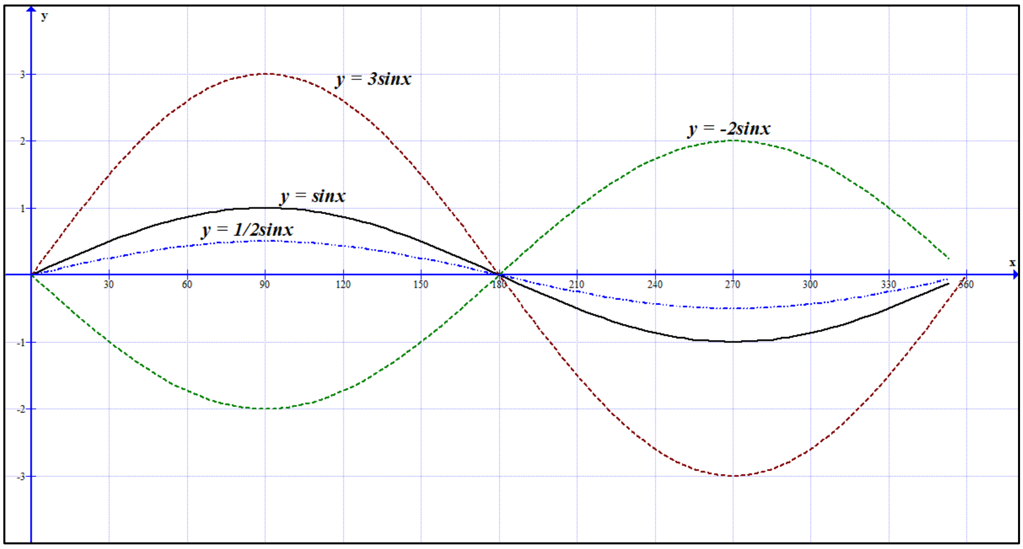
***a*-amplitude**. For the basic graph $a=1$

The amplitude (a-value) is the coefficient of the function. Positive a-value is the minimum value while the negative a-value is the maximum value of the function.

The graph is stretched or compressed vertically

The bigger the value of ***a***, the bigger the maximum value will be.

Changing ***a*** doesn't change the x-intercepts when ***q*** = 0.



ACTIVITY 2:

Complete the table and draw the following graphs on the same set of axis:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y=\sin x$													
$y=\sin\frac{1}{2}x$													
$y=\sin 3x$													
$y=\cos x$													
$y=\cos\frac{1}{2}x$													
$y=\cos 3x$													
$y=\tan x$													
$y=\tan\frac{1}{2}x$													
$y=\tan 3x$													

2.1 Investigate 'k' in the graphs:

$y=\sin x$

$y=\sin 3x$

$y=\sin\frac{1}{2}x$

2.2 Investigate 'k' in the graphs:

$y=\cos x$

$y=\cos 3x$

$y = \cos\frac{1}{2}x$

2.3 Investigate '*k*' in the graphs:

$y = \tan x$

$y = \tan 3x$

$y = \tan\frac{1}{2}x$

GROUP DISCUSSIONS

Changing the period: [The effects of *k*]

$y = a\sin(kx)$, $y = a\cos(kx)$, $y = \tan(kx)$

The change in period stretches or compresses the graph horizontally. In the case of a change in period, the angle will be multiplied by a factor of ***k***.

- If $0 < k < 1$ the graph is stretched and the period increases.
- If $k > 1$ the graph is compressed and the period decreases.

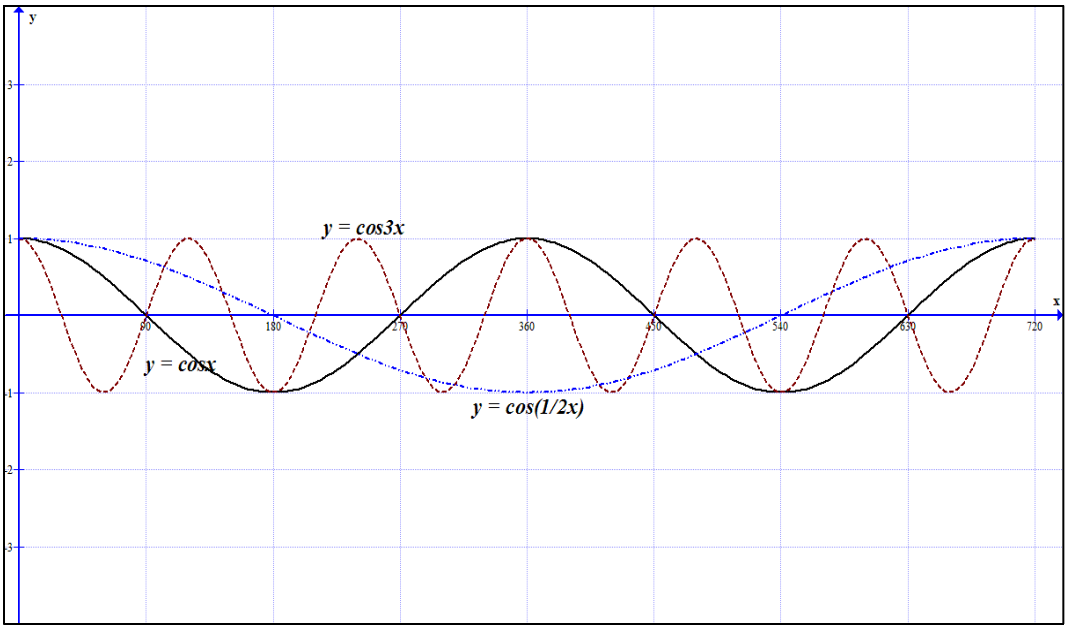
Period of a graph is the interval taken for one complete one cycle of its basic shape.
The value of ***k*** represents the frequency of the graph, i.e., the number of curves completed in one "***normal***" period. Period of the graph is affected by the value of ***k.***
To determine the period, ***k*** is divided into the "***normal***" period.

$$For\ \ sin e\ and\ cosine : Period = \frac{360°}{k}$$

$$For\ \ tan\ gent : Periond = \frac{180°}{k}$$

Period of $180°$, it means the graph repeats itself every $180°$

For the graph of $y = \cos\frac{1}{2}x$, $Period = \frac{360°}{\frac{1}{2}} = 720°$, it means the graph repeats itself every $720°$



ACTIVITY 3:

Complete the table and draw the following graphs on the same set of axis:

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$y=\sin x$													
$y=\sin(x-30°$													
$y=\sin(x+60°$													
$y=\cos x$													
$y=\cos(x-30°)$													
$y=\cos(x+60°)$													
$y=\tan x$													
$y=\tan(x-30°)$													
$y=\tan(x+45°)$													

3.1 Investigate '*p*' in the graphs:

$y=\sin x$

$y=\sin(x-30°)$

$y=\sin(x+60°)$

3.2 Investigate '*p*' in the graphs:

$y=\cos x$

$y=\cos(x-30°)$

$y=\cos(x+60°)$

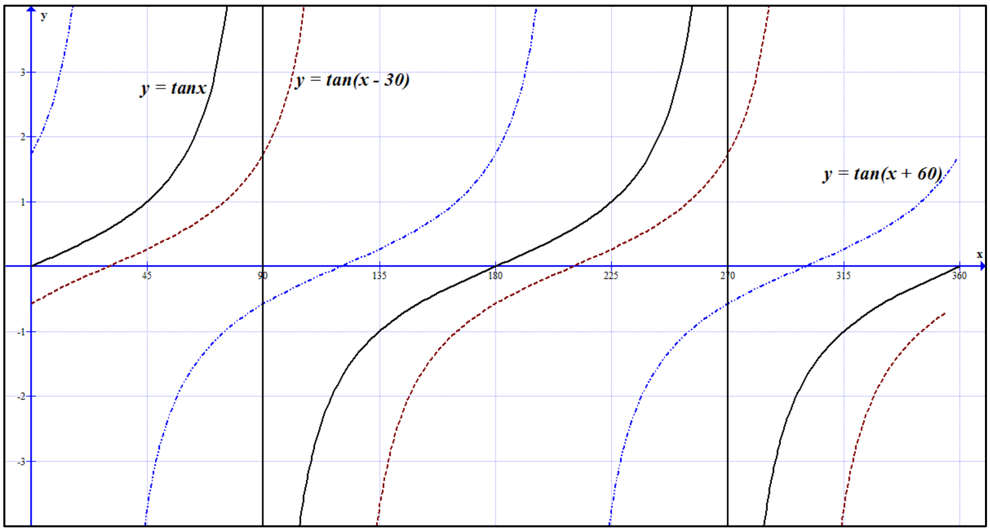
3.3 Investigate '*p*' in the graphs:

$y = \tan x$

$y = \tan(x - 30°)$

$y = \tan(x + 45°)$

GROUP DISCUSSIONS



Similar principles applied on sine graph are also applicable to cosine and tangent graphs!!

Horizontal shifting (Moving the *y*-axis): [The effects of *p*]

$y = a\sin(x \pm p)$, $y = a\cos(x \pm p)$, $y = \tan(x \pm p)$

Moving the *y*-axis: the graph is shifting to the left or to the right.

When moving the *y*-axis, we change the *x*-value. If the *y*-value is moved **forward (to the right)**, we **add** the distance moved to *x*. For example, start with *y* = sin*x*. The *y*-axis moved $30°$ to the right gives $y = \sin(x + 30°)$

ACTIVITY 4:

Complete the table and draw the following graphs on the same set of axis:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y=\sin x$													
$y=1+\sin x$													
$y=\sin x-2$													
$y=\cos x$													
$y=1+\cos x$													
$y=\cos x-2$													
$y=\tan x$													
$y=1+\tan x$													
$y=\tan x-2$													

4.1 **Investigate '*q*' in the graphs**:

$y=\sin x$

$y=1+\sin x$

$y=\sin x-2$

4.2 Investigate '*q*' in the graphs:

$y=\cos x$

$y=1+\cos x$

$y=\cos x-2$

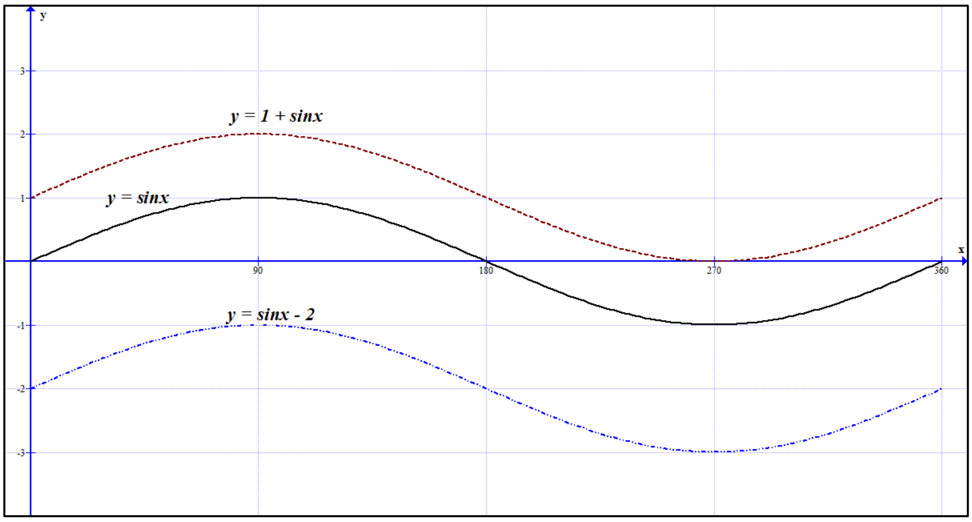
4.3 Investigate '*q*' in the graphs:

$y=\tan x$

$y=1+\tan x$

$y=\tan x-2$

GROUP DISCUSSIONS



Vertical shifting (Moving the x-axis): [The effects of *q*]

$y = a\sin\theta \pm q$, $y = a\cos\theta \pm q$, $y = a\tan\theta \pm q$

Moving the *x*-axis: the graph is shifting upwards or downwards

- If ***q*** is a**dded** to the graph, the graph is shifted ***q*** units **upwards**
- If ***q*** is **subtracted** to the graph, the graph is shifted ***q*** units **downwards**

- If you move the *x*-axis you are changing the y-value; i.e., If you move the *x*-axis up by 1 unit, you are adding 1 to the y-value. For example, y = sin*x*. Move *x*-axis up by 1 unit. The equation becomes:

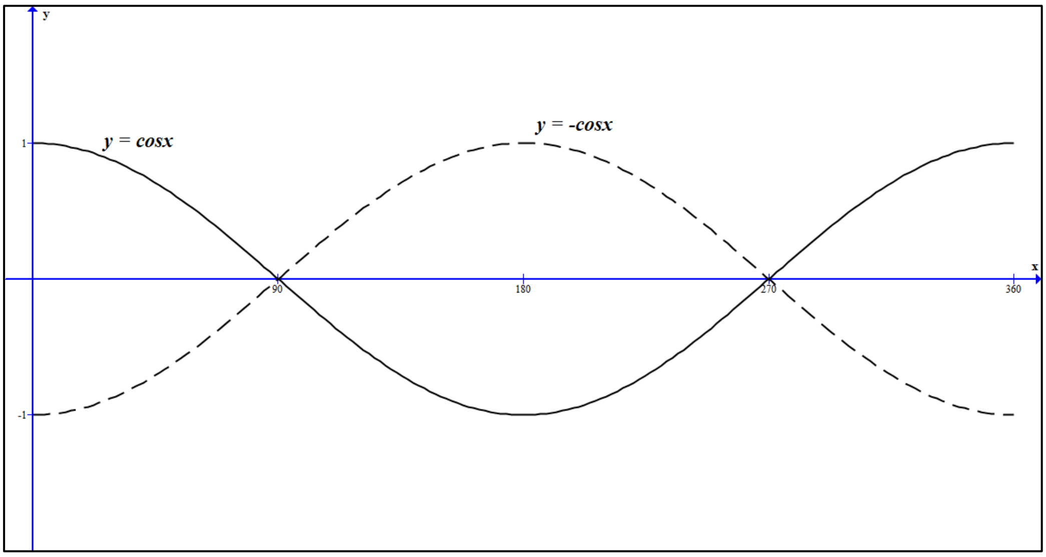
  $y+1=\sin x$

  $y=\sin x-1$

q*-rest position**. For the basic function ***q = 0, this value shifts the whole graph vertically up when it is positive and down when it is negative. The ***q-value*** changes the position of the rest position and will change the value of the intercepts.

Reflections about the *x*-axis

The *x*-axis is the axis of symmetry, i.e., $y = 0$. If $(x; y)$ is reflected about the *x*-axis, the coordinates are $(x; -y)$. This affects the *y*-value in the graph equation, which becomes negative.



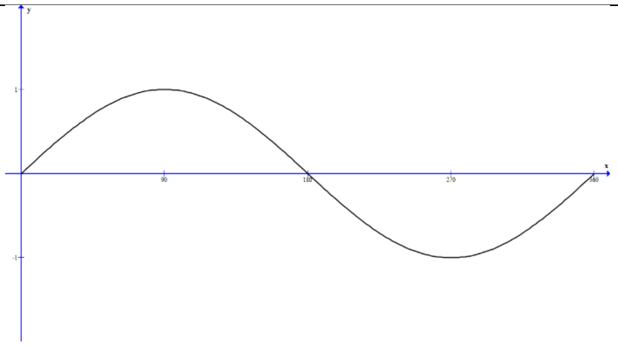
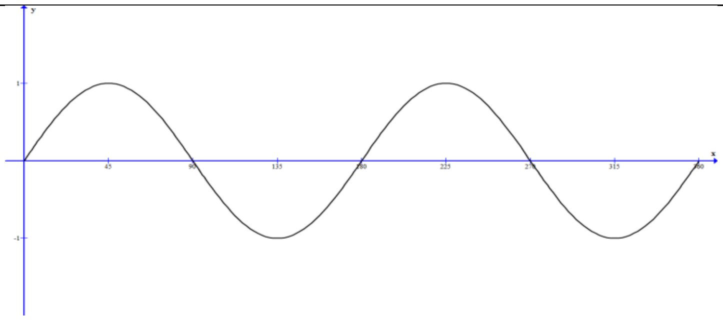
Summary on Translations (moving the axes)

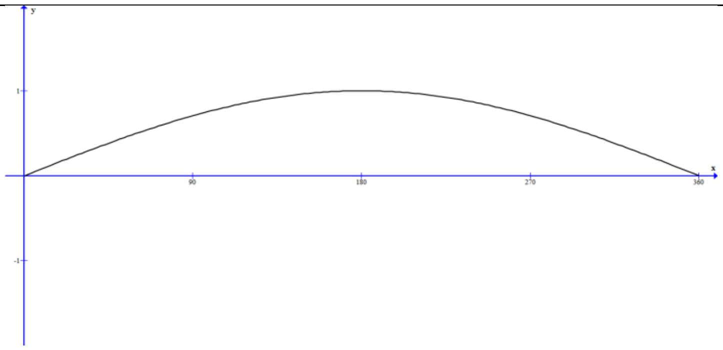
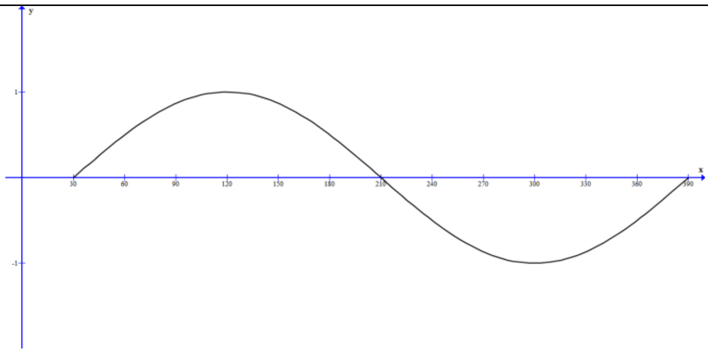
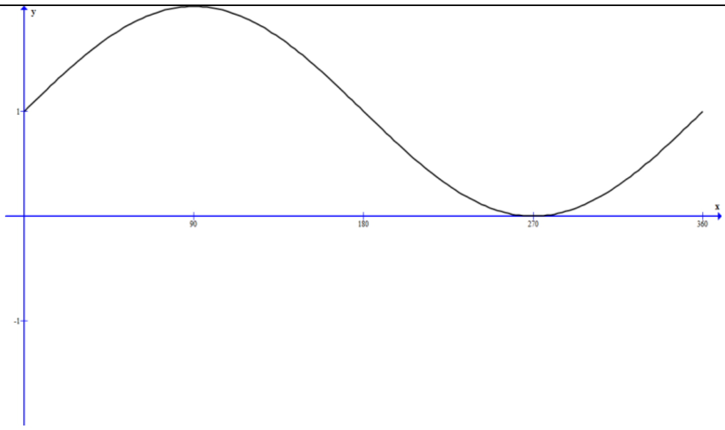
Function	Translation	Amount of shift	Vertical or horizontal direction	Example
$y=\sin(kx-p)$	**Horizontal**	$kx-p=0; x=\frac{p}{k}$	**Right**	$y=\sin(x-60°)$
$y=a\sin x+q$	**Vertical**	q	**Top (Upward)**	$y=2+\sin x$
$y=\sin(kx-p)$	**Horizontal**	$kx+p=0; x=-\frac{p}{k}$	**Left**	$y=\sin(x+60°)$
$y=a\sin x-q$	**Vertical**	$-q$	**Down (Downward)**	$y=-3+\sin x$

Shrinking and Stretching

Where does it happen?	**For** $y=a\sin bx$	**Period**	$y=a\cos bx$	**Period**
Stretching **Example of a function**	$0<b<1$ $y=\sin\frac{1}{2}x$	$\frac{360°}{\lvert b\rvert}$ $\frac{360°}{\frac{1}{2}}=720°$	$0<b<1$ $y=\cos\frac{x}{3}$	$\frac{360°}{\lvert b\rvert}$ $\frac{360°}{\frac{1}{3}}=1080°$
Shrinking **Example of a function**	$b>1$ $y=\sin 2x$	$\frac{360°}{\lvert b\rvert}$ $\frac{360°}{2}=180°$	$b>1$ $y=\sin 3x$	$\frac{360°}{\lvert b\rvert}$ $\frac{360°}{3}=120°$

Examples of translation, shrinking and stretching

Equation	**Graph**	**Effect**
$y=\sin\theta$		Basic
$y=\sin 2\theta$		Shrinking $Period=\frac{360°}{2}=180°$

$y=\sin\frac{1}{2}\theta$		Stretching $Period=\frac{360°}{\frac{1}{2}}=720°$
$y=\sin(\theta-30°)$		Translation Horizontal shift: $30°$ to the right
$y=\sin\theta+1$		Translation Vertical shift 1 unit upwards

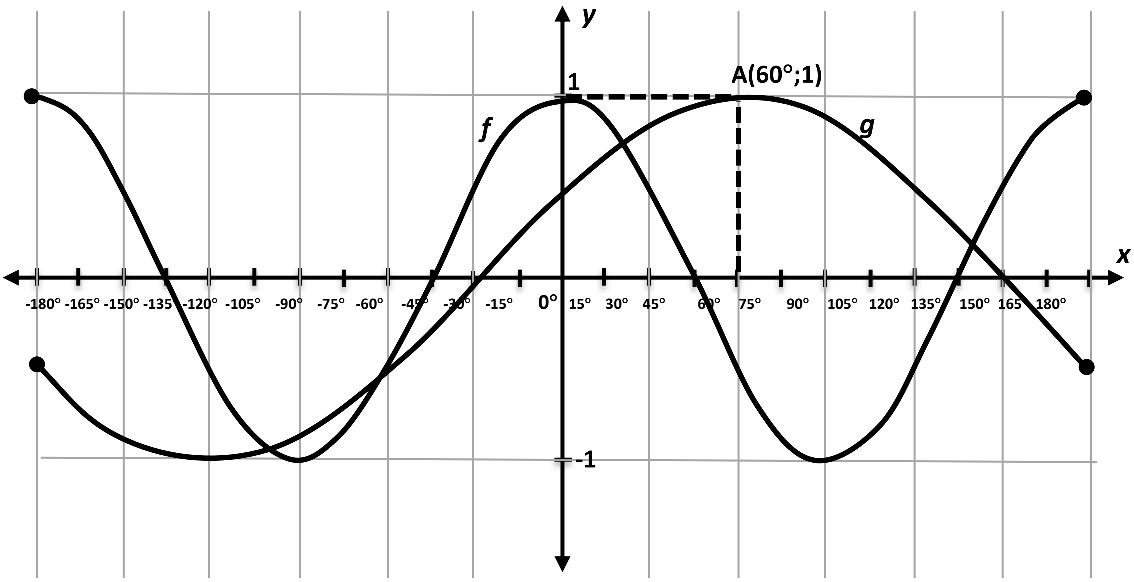
Strategies for Graphical Interpretations: Typical interpretation questions

Type of question	Interpretation
$f(x) < 0$	$f(x)$ must be less than zero $f(x)$ lies below the x-axis value is excluded
$f(x) < g(x)$	$f(x)$ must be less than $g(x)$ $f(x)$ lies below $g(x)$ e.g. $\sin(x)$ lies below $\cos(x)$
$f(x) \geq 0$	$f(x)$ must be greater than or equal to zero $f(x)$ lies above x-axis value is included
$f(x) > g(x)$	$f(x)$ must be greater than $g(x)$ $f(x)$ lies above $g(x)$ value is excluded
$f(x).\ g(x) < 0$	One of the two graphs must be above x-axis while the other must be below x-axis. Value is excluded
$f(x).g(x) \geq 0$	Both the two graphs are above the x-axis or both the two graphs are below the x-axis. Value is included.
$f(x)\ .\ g(x) = 0$	Either $f(x)=0$ or $g(x)=0$ x-intercepts of f(x) and g(x)
$f(x) - g(x) = 0$	$f(x) = g(x)$ Points of intersection
(	Excluded because of restriction
]	Included because of restriction
(;]	*first*-value excluded and *last*-value is included because of restriction
(;) or $<$ $>$ or ○	Values excluded
[;] or $\geq$ $\leq$ or •	Values included

Examples:

QUESTION 1

The sketch below shows the graphs of $f(x) = \cos 2x$ and $g(x) = \sin(x - \theta)$ for $x \in [-180° ; 180°]$. $A(60° ; 1)$ is a point on the graph of g. Use the graph to answer the questions that follow.



1.1 Write down the value of θ. (1)

1.2 Determine the period of f. (1)

1.3 If $h(x) = f(x) - 1$, write down the range of h. (1)

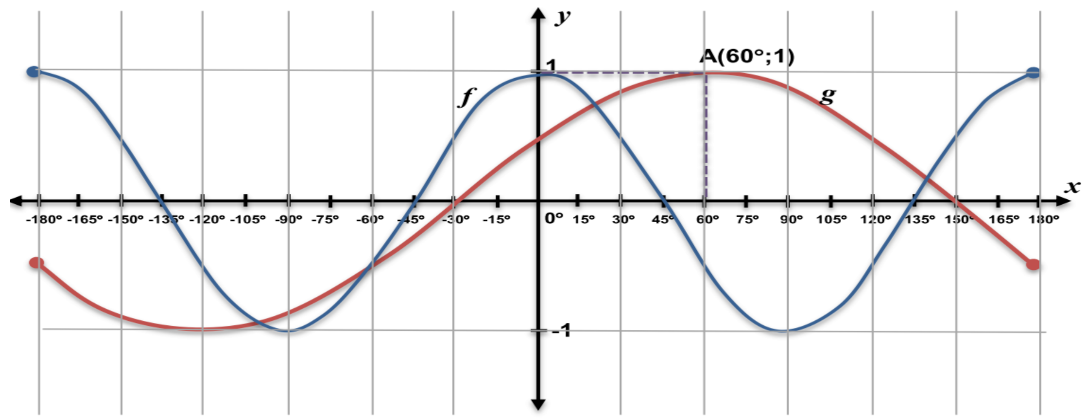
1.4 Determine the values of x, where $x \in [0° ; 180°]$ for which

1.4.1 $f(x).g(x) < 0.$ (3)

1.4.2 $f'(x).g(x) > 0.$ (2)

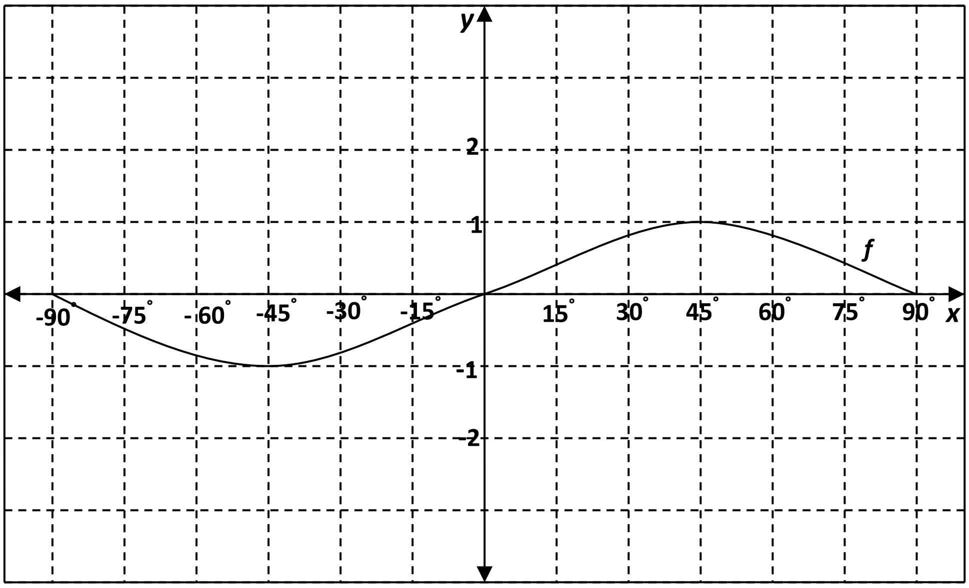
[8]

SOLUTIONS

		
1.1	$\sin(60° - \theta) = 1$ $60° - \theta = 90°$ $\theta = -30°$	✓ $\theta = -30°$ (1)
1.2	$\text{period / periode} = \dfrac{360°}{2}$ $= 180°$	✓ $180°$ (1)
1.3	$-2 \leq y \leq 0$ of / or $[-2;0]$	✓ $-2 \leq y \leq 0$ (1)
1.4.1	$45° < x < 135°$ or/of $150° < x \leq 180°$	✓ $45°$ and $135°$ ✓ $150°$ and $180°$ ✓ ALL inequalities correct ***Subtract 1 mark for extra intervals*** (3)
1.4.2	$90° < x < 150°$	✓ $90°$ and/en $150°$ ✓ inequalities (2) **[8]**

QUESTION 2

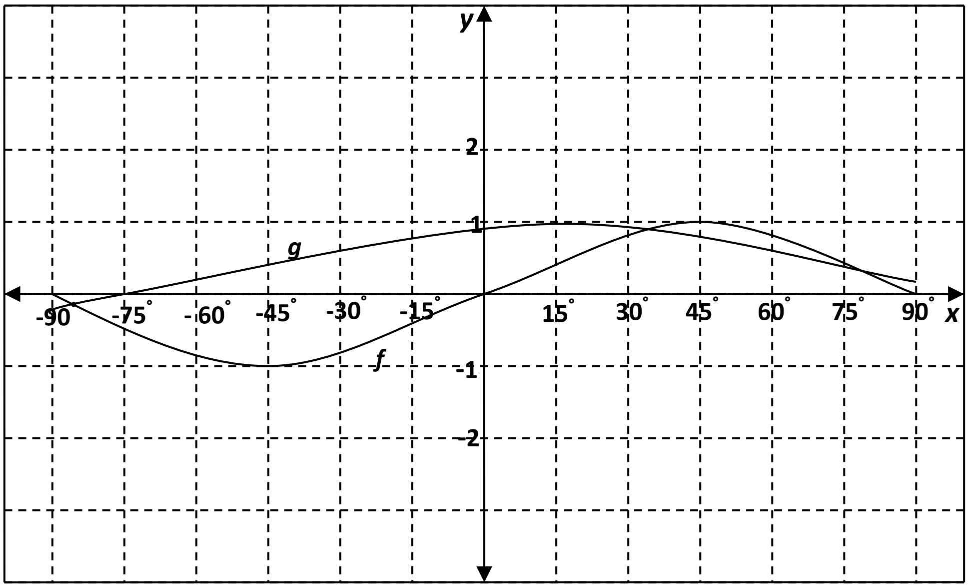
Consider the function *f(x)* = sin2*x* for $x \in [-90°;\ 90°]$



2.1	Write down the period of *f*.	(1)
2.2	Sketch the graph of $g(x) = \cos(x - 15°)$ *for* $x \in [-90°\ ;\ 90°]$ on the diagram sheet provided for this sub-question.	(5)
2.3	Find the values of *x* for which *f(x)* < *g(x)*.	(3)
		[9]

SOLUTION Q.2

2.1	Period of *f* =180°	✓180° (1)

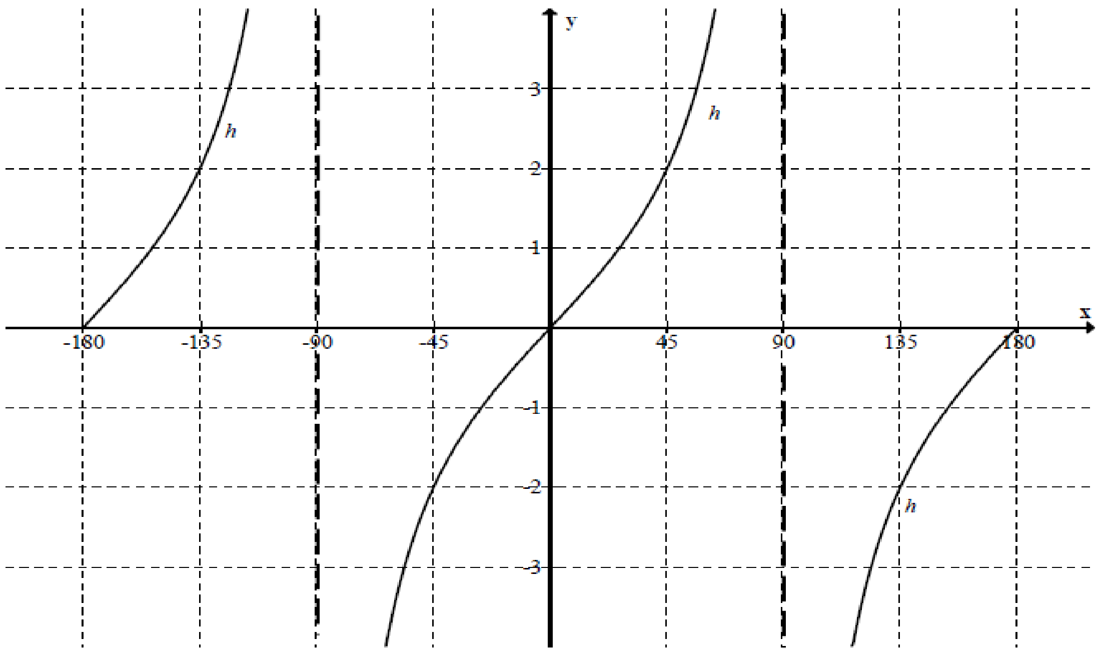
<table>
<tr><td>2.2</td><td colspan="2">

$g\ (x)$: ✓ x -intercept
✓ y -intercept
✓ Turning point
✓✓ Endpoint
(5)</td></tr>
<tr><td>2.3</td><td>$-85° < x < 35°$ or $x > 75°$</td><td>✓✓Critical values
✓Notation
(3)</td></tr>
<tr><td></td><td></td><td>[9]</td></tr>
</table>

TYPICAL EXAM QUESTIONS AND COGNITIVE LEVELS AND EXTENDED OPPORTUNITIES

QUESTION 1

The graph of $h(x) = a\tan x$; for $x \in [-180°; 180°]$, $x \neq -90°$, is sketched below.



1.1 Determine the value of *a*. (2)

1.2 If $f(x) = \cos(x + 45°)$, sketch the graph of f for $x \in [-180°; 180°]$, on the diagram provided in your ANSWER BOOK. (4)

1.3 How many solutions does the equation $h(x) = f(x)$ have in the domain $[-180°; 180°]$? (1)

[7]

QUESTION 2

2.1 On the same system of axes, sketch the graphs of $f(x) = 3\cos x$ and $g(x) = \tan\frac{1}{2}x$ for $-180° \leq x \leq 360°$. Clearly show the intercepts with the axes and all turning points. (5)

Use the graphs in 2.1 to answer the following questions.

2.2 Determine the period of *g*. (1)

2.3 Determine the co-ordinates of the turning points of f on the given interval. (2)

2.4 For which values of x will both functions increase as x increases for $-180° \leq x \leq 360°$? 2)

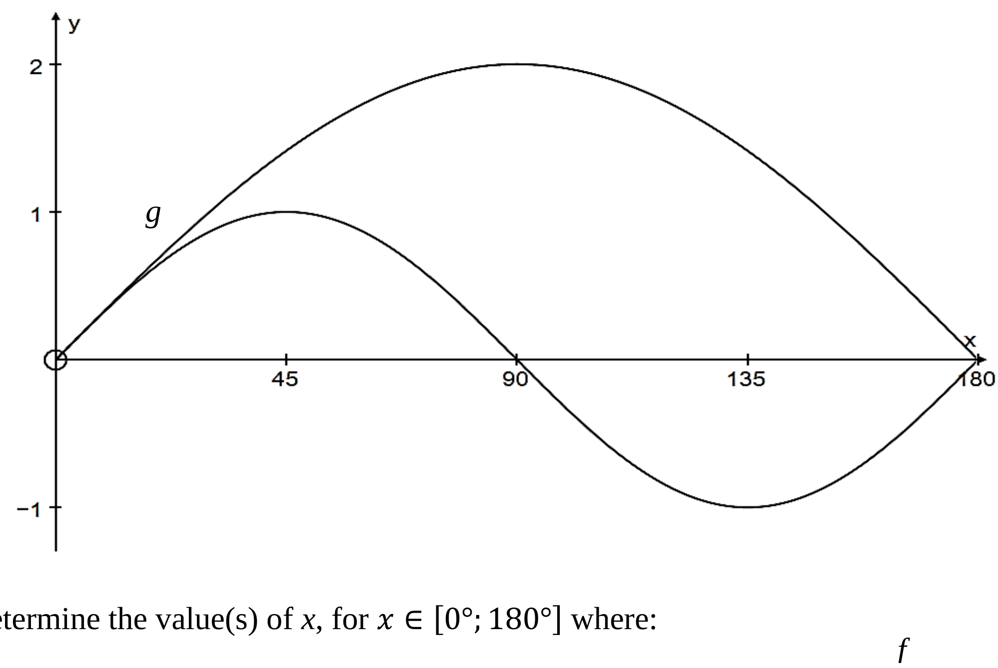
2.5 If the y–axis is moved $45°$ to the left, then write down the new equation of f in the form $y = \ldots..$ (1)

[11]

QUESTION 3

The graphs below represent the functions of f and g.

$f(x) = \sin 2x$ and $g(x) = c\sin dx$, $x \in [0°; 180°]$



3.1 Determine the value(s) of x, for $x \in [0°; 180°]$ where:

3.1.1 $g(x) - f(x) = 2$ (1)

3.1.2 $f(x) \leq 0$ (2)

3.1.3 $g(x).f(x) \geq 0$ (3)

3.2 f in the graph drawn above undergoes transformations to result in g and h as given below. Determine the values of a, b, c and d if

3.2.1 $g(x) = c\sin dx$ (2)

5.2.2 $h(x) = a\cos(x - b)$ (2)

[10]

QUESTION 4

4.1 Given the equation $2\cos\theta = \sin(\theta + 30°)$.

4.1.1 Show that $2\cos\theta = \sin(\theta + 30°)$ is equivalent to $\sqrt{3}\sin\theta = 3\cos\theta$ (3)

4.1.2 Hence, or otherwise, calculate θ if $\theta \in [-180°; 180°]$

4.2 Consider the functions $f(\theta) = 2\cos\theta$ and $g(x) = \sin(\theta + 30°)$ for $\theta \in [-180°; 180°]$:

4.2.1 Sketch graphs of f and g on the same set of axes.
Show intercepts on the axes clearly. (5)

4.2.2 Use the graphs to determine $\theta \in [-90°; 90°]$ if $f(\theta).g(\theta) > 0$ (3)

4.2.3 For which values of θ will $g^{'}(x) > 0]$, $\theta \in [-180°; 180°]$? (3)

[18]

QUESTION 5

Consider : $f(x) = \cos(x - 45°)$ and $g(x) = \tan \frac{1}{2}x$ for $x \in [-180° ; 180°]$

5.1 Use the grid provided to draw sketch graphs of f and g on the same set of axes for $x \in [-180° ; 180°]$. Show clearly all the intercepts on the axes, the coordinates of the turning points and the asymptotes. (6)

5.2 Use your graphs to answer the following questions for $x \in [-180° ; 180°]$

5.2.1 Write down the solutions of $\cos(x - 45°) = 0$ (2)

5.2.2 Write down the equations of asymptote(s) of g. (2)

5.2.3 Write down the range of f. (1)

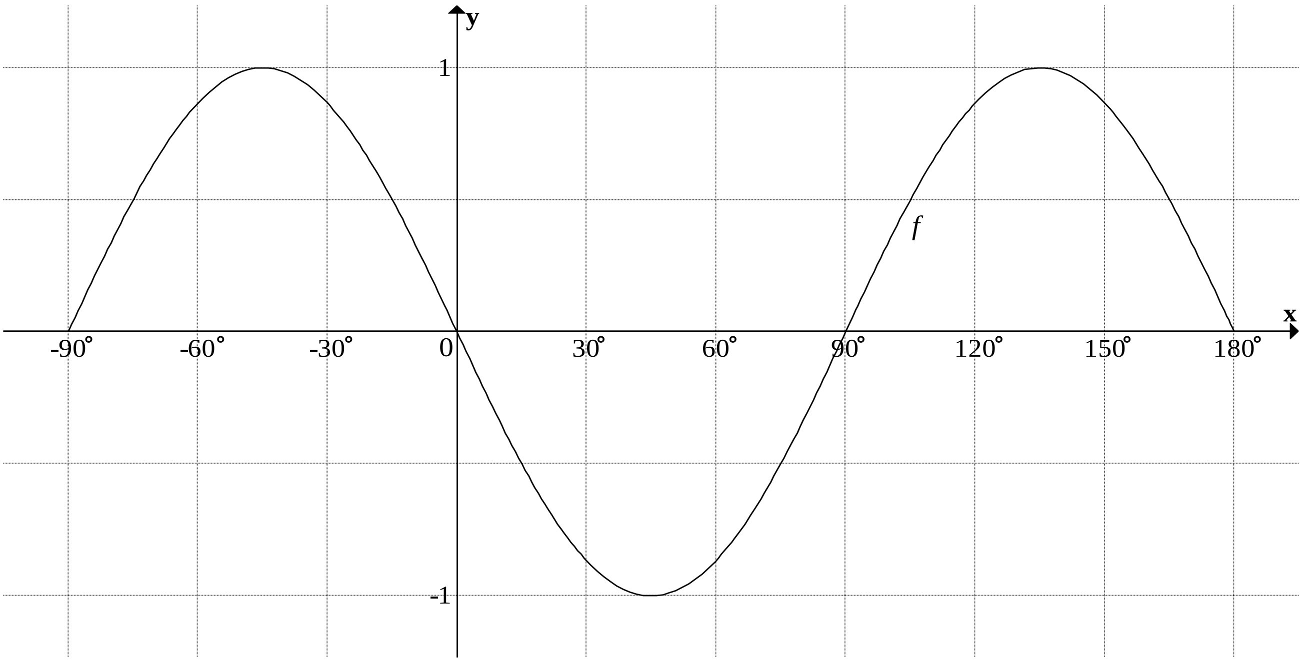
5.2.4 How many solutions exist for the equation $\cos(x - 45°) = \tan \frac{1}{2}x$? (1)

5.2.5 For what value(s) of x *is* $f(x).g(x) > 0$ (3)

[15]

QUESTION 6

In the diagram, the graph of $f(x) = -\sin 2x$ is drawn for the interval $x \in [-90°; 180°]$.



6.1. Draw the graph of g, where $g(x) = \cos(x - 60°)$ on the same system of axes for the interval $x \in [-90°; 180°]$. (3)

6.2 Use your graphs to solve x if $f(x) \leq g(x)$ for $x \in [-90°; 180°]$ (3)

6.3 If the graph of f is shifted $30°$ left, give the equation of the new graph which is formed (2)

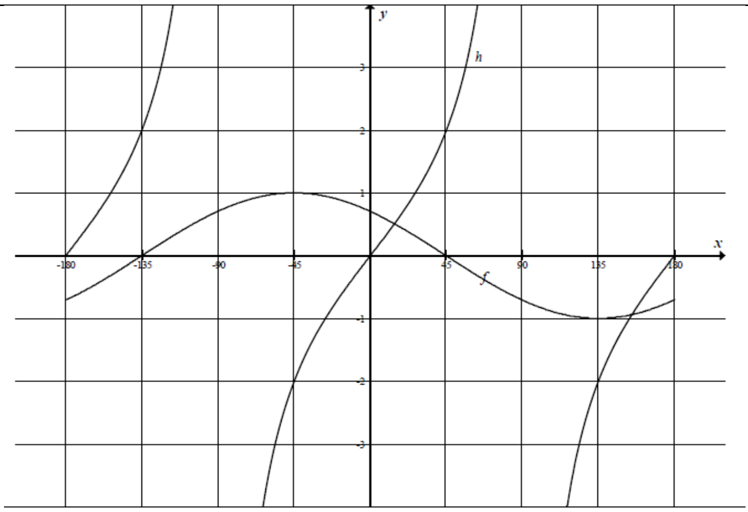
6.4 What transformation must the graph of g undergo to form the graph of h, where $h(x) = \sin x$? (2)

[10]

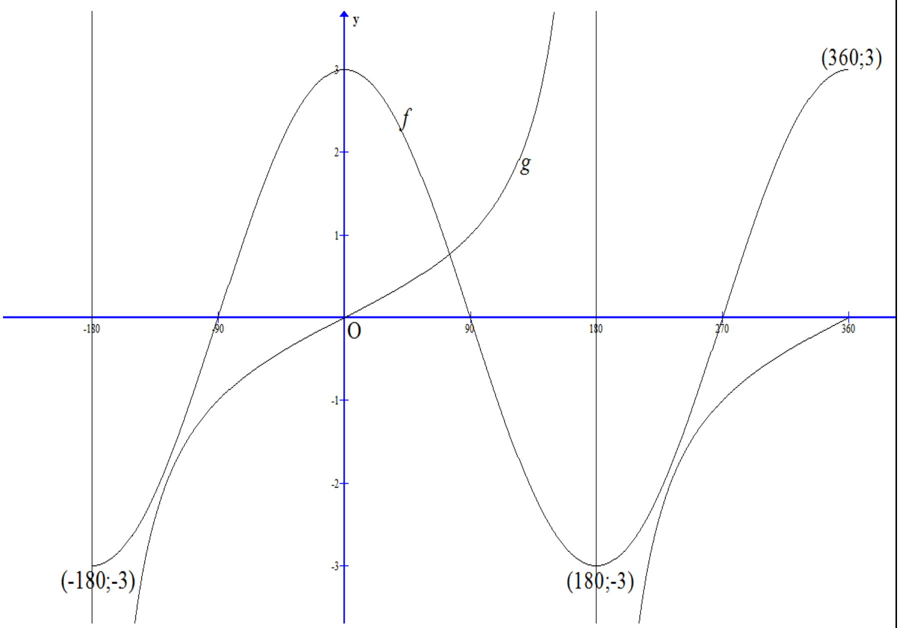
SOLUTIONS

QUESTION 1

1.1	$a \tan 45° = 2$ $\therefore a = 2$	✓Reading from graph ✓Answer (2)

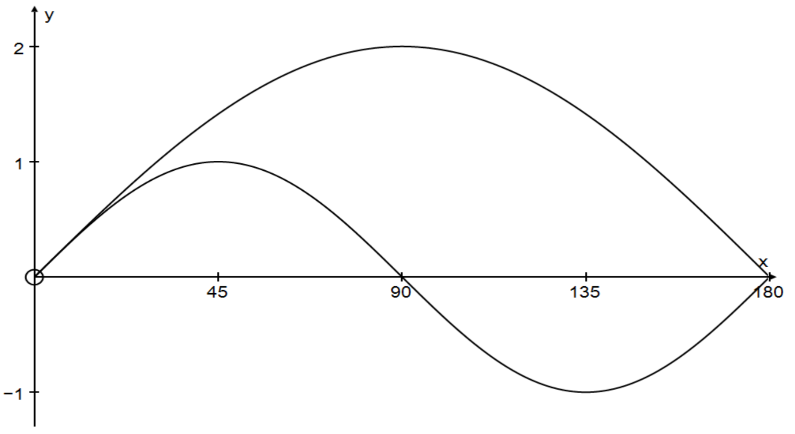
1.2		✓Amplitude ✓Shape ✓✓Turning points clearly indicated as $(-45°, 1)$ and $(135°, -1)$ (4)
1.3	2 solutions.	✓2 solutions (1)
		[7]

QUESTION 2

2.1 	A✓asymptotes A✓shape of f A✓shape of g A✓correct x- intercept of f (-90^o; 90^o; 270^o) A✓correct x – intercept of g (0^o; 360^o) (5)
2.2 360°	A✓360°

		(1)
2.3	(0; 3) and (180° ; – 3) (–180° ; –3) and (360° ; 3)	CA✓for any two (2)
2.4	–180º < x < 0° or 180° < x < 360° OR –180º < x < 0º ⋃ 180º < x < 360º	CA✓–180° < x < 0° CA✓ 180 < x < 360° [penalize one mark for incorrect notation] (2)
2.5	y = 3cos (x – 45°)	A✓3cos (x – 45°) (1) **[11]**

QUESTION 3



3.1.1	$x = 90°$	✓ (1)
3.1.2	$x \in [90°; 180°]$	✓critical points ✓ notation (2)
3.1.3	$x \in [0°; 90°] \cup x = 180°$	✓ ✓$[0°; 90°] \cup$✓$180°$ (3)
3.2.1	$g(x) = 2sinx$ c = 2 d=1	✓c = 2✓ d=1 (2)
3.2.2	$h(x) = 2cos(x - 90°)$ a = 2 b = 90°	✓a = 2 ✓ b = 90° (2)
		[10]

QUESTION 4

<table>
<tr>
<td>4.1.1</td>
<td>

$2\cos\theta = \sin(\theta + 30^\circ)$

$= \sin\theta\cos 30^\circ + \cos\theta\sin 30^\circ$

$= \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$

$\therefore 4\cos\theta = \sqrt{3}\sin\theta + \cos\theta$

$\therefore 3\cos\theta = \sqrt{3}\sin\theta$

</td>
<td>

✓ expansion

✓ substitution of special angles

✓ $4\cos\theta = \sqrt{3}\sin\theta + \cos\theta$

(3)

</td>
</tr>
<tr>
<td>4.1.2</td>
<td>

$3\cos\theta = \sqrt{3}\sin\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{3}{\sqrt{3}}$

$\tan\theta = \frac{3}{\sqrt{3}}$

$\theta = 60^\circ + 180.n$

$\theta = -120^\circ \; or \; \theta = 60^\circ$

</td>
<td>

✓ division

✓ $\tan\theta = \frac{3}{\sqrt{3}}$

✓✓ answers

(4)

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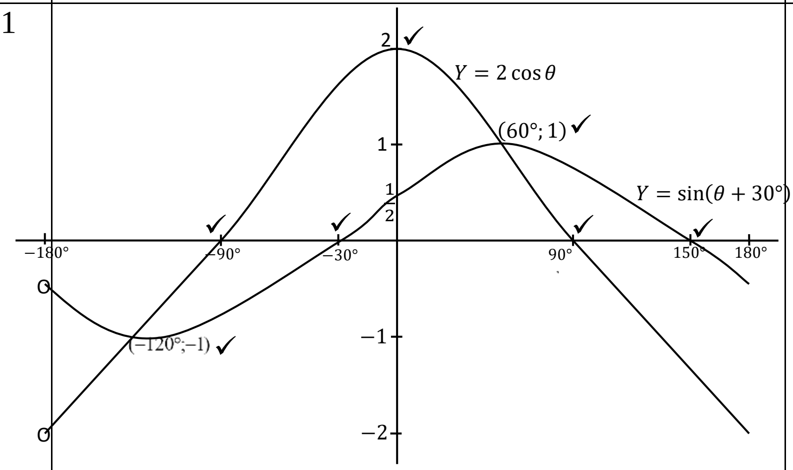
<tr>
<td>4.2.1</td>
<td>



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<td>

g :

✓ y-intercept and x- intercepts

✓ shape

f :

✓ y-intercept and x-intercepts

✓ turning points

✓ shape

(5)

</td>
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<tr>
<td>4.2.2</td>
<td>

$-30^\circ < \theta < 90^\circ$

</td>
<td>

✓✓ values

✓ notation (3)

</td>
</tr>
<tr>
<td>4.2.3</td>
<td>

$-120^\circ < \theta < 60^\circ$

</td>
<td>

✓✓ values

✓ notation

(3)

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<td>

[18]

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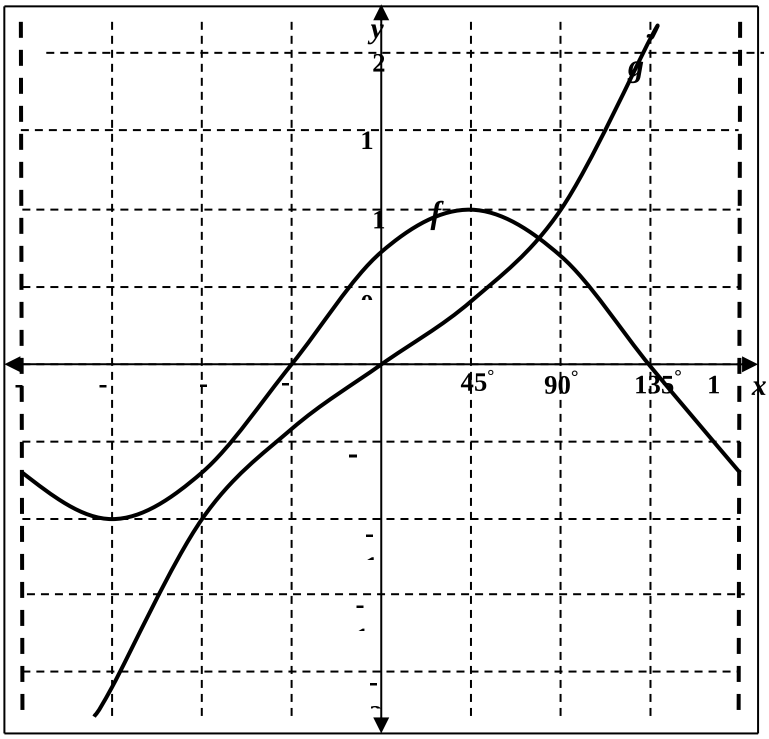
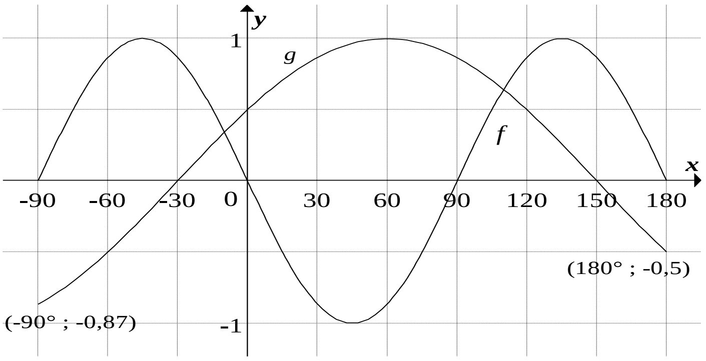
QUESTION 5

5.1	 *f* : ✓ *x*-int ✓ *y*-int, ✓turning points *g*: ✓asymptotes ✓ passing (0,0), ✓ (90;1) (6)	
5.2.1	$x = -45°$ or $x = 135°$	✓ $x = -45°$ ✓ $x = 135°$ (2)
5.2.2	$x = 180°$ *or / of* $x = -180°$	✓$x = 180°$ ✓ $x = -180°$ (2)
5.2.3	$y \in [-1;1]$ or/*of* $-1 \leq y \leq 1$	✓answer (1)
5.2.4	1	✓answer (1)
5.2.5	$x \in (-180°; -45°)$ *or / of* $(0°\ ;135°)$ **OR** $-180° < x < -45°$ or/of $0° < x < 135°$	✓ – 180°; – 45° ✓ 0°; 135° ✓notation (3) **[15]**

QUESTION 6

Q6	**SUGGESTED ANSWER**	**DESCRIPTORS/**	MARK
6.1		✓ shape ✓ endpoints ✓ intercepts with axes	(3)
6.2	$-10° \leq x \leq 110°$	✓ Inequality ✓✓Ans	(3)
6.3	$y = -\sin(2x + 30°)$	✓ ✓ Ans	(2)
6.4	g must shift 30° right	✓ 30° ✓ right	(2)
			[10]

Where are we now?

Module 1	Module 2	Module 3	Module 4
Algebraic functions & graphs	*Cubic functions*	*Trigonometric Graphs*	*Financial Mathematics*

FINANCIAL MATHEMATICS

Introduction

One of the most common misconceptions found in the Grade 12 examinations is the lack of understanding that learners have from the previous grades (Grades 10 and 11) and the lack of ability to manipulate the formulae.

In addition to this, many learners do not know when to use which formulae, or which value should be allocated to which variable.

Overall aim of the module: Financial Mathematics

The study of Financial Mathematics is centred on the concepts of simple and compound growth. The learner must be made to understand the difference between the two concepts at Grade 10 level. This may then be successfully built upon in Grade 11, eventually culminating in the concepts of Present and Future Value Annuities in Grade 12. This section will revise the most important concepts that lead up to Present and Future Value Annuities.

Learning objectives of the module

- To understand and become proficient in the use of different financial mathematics formulae.
- To understand the difference between present value and future value.
- To understand the difference between simple interest growth and compound interest growth
- To use simple $\{A = P(1 - in)\}$ and compound decay $\{A = P(1 - i)^n\}$ to solve problems (including straight line depreciation and depreciation on a reducing balance. Link to work on functions.
- To understand the effects of different periods of compounding growth and decay (including effective and nominal interest rates).
- To apply the knowledge of geometric series to solve annuity and bond repayment.
- To critically analyse different loan options.

Key Content Addressed by this module: ATP (Show curriculum mapping)

CAPS extraction indicating progression from Grades 10-12.

GRADE 10	GRADE 11	GRADE 12
Use simple $\{A = P(1 + in)\}$ and compound growth $\{A = P(1 + i)^n\}$ formulae to solve problems (including interest, hire purchase, inflation, population growth and other real-life problems). The implications of fluctuating foreign exchange.	Use simple $\{A = P(1 - in)\}$ and compound decay $\{A = P(1 - i)^n\}$ to solve problems (including straight line depreciation and depreciation on a reducing balance. Link to work on functions. The effects of different periods of compounding growth and decay (including effective and nominal interest rates).	Calculate the value of $\boldsymbol{n}$ in the formulae $A = P(1 \pm i)^n$ Apply the knowledge of geometric series to solve annuity and bond repayment. Critically analyse different loan options.

WEIGHTINGS OF CONTENT AREAS			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Finance and growth	$\mathbf{10 \pm 3}$		
Finance, growth and decay		$\mathbf{15 \pm 3}$	$\mathbf{15 \pm 3}$

TERMS	GRADE 10		GRADE 11		GRADE 12	
	Topic	**No of weeks**	**Topic**	**No of weeks**	**Topic**	**No of weeks**
TERM 1					Finance, growth and decay	(2 Weeks)
TERM 3	Finance and growth	Two weeks (2 Weeks)	Finance, growth and decay	(2 weeks)		

Activities/ Tasks

Format: Questions, answers and discussion.

SIMPLE AND COMPOUND GROWTH **[Duration:40 Minutes]**

- What is our understanding of simple and compound growth?
- How do we, as educators, effectively transfer our understanding of these concepts to our learners?
- What do the learners need to know before we can begin to explain the difference between simple and compound growth?

Example:

Tau invests R1 000 in a savings plan for a period of 5 years. He will receive 12% interest per annum on his savings.

Calculate the interest accrued in 5 years using both **SIMPLE** and **COMPOUND GROWTH** plans.

SIMPLE GROWTH (Straight line method)

Interest is calculated at the start of the investment based on the money he is investing and **WILL REMAIN THE SAME** every year of his investment.

Solution:

$$Interest = P\left(\frac{i}{100}\right)$$

$$= 1000\left(\frac{12}{100}\right)$$

= R120 per year

$$A = P(1 + in)$$

$$A = 1\ 000(1 + 0{,}12 \times 5)$$

= R1 600.

IN A TABLE

Year 1	**Year 2**	**Year 3**	**Year4**	**Year 5**
1 000 +120	1 120+120	1 240+120	1 360+120	1 480+120
1 120	1 240	1 360	1 480	1 600

Tau will earn an interest of R600 in 5 years. Tau will receive R1 600 at the end of plan.

COMPOUND GROWTH /DECAY

The Compound growth plan has interest that is recalculated every year based on the money that is in the account. The interest **WILL CHANGE** every year of the investment.

Year 1	Year 2	Year 3	Year 4	Year 5
$Interest = p\left(\frac{i}{100}\right)$	$Interest = p\left(\frac{i}{100}\right)$	$Interest = p\left(\frac{i}{100}\right)$	$Interest = p\left(\frac{i}{100}\right)$	$Interest = p\left(\frac{i}{100}\right)$
$Interest = 1000\left(\frac{12}{100}\right)$	$Interest = 1120\left(\frac{12}{100}\right)$	$Interest = 1254.4\left(\frac{12}{100}\right)$	$Interest = 1404{,}93\left(\frac{12}{100}\right)$	$Interest = 1404{,}93\left(\frac{12}{100}\right)$
$Interest = 120$	$Interest = 134{,}4$	$Interest = 150{,}53$	$Interest = 168{,}59$	$Interest = 188{,}76$
R1 120	R1 254.4	R1 404,93	R1 572,99	R1 761,75

Tau will earn an interest of R761,75 in 5-year period. Interest earned per year through compound growth plan is not constant as in simple growth plan.

The process could be summed up as:

$$A = P\left(1 + \frac{i}{100}\right)^n$$

$$A = 1000\left(1 + \frac{12}{100}\right)^5$$

$$A = R1\,761{,}74c$$

ACTIVITY 1 **[Duration: 30 min]**

1. Thato invest a R25 000 in a savings account for 4 years. The investment earns an interest of 8,5% p.a, on a straight-line method.
   1.1.1 Calculate the interest to be earned by Thato after 4 years?
   1.1.2 Calculate the amount in Thato's savings account at the end of 4 years.

   Solutions
   1.1 $interest = Pin = 25000 \times 0{,}085 \times 4 = 8500$
   1.2 A = 25 000+8 500= R33 500

2. How long would the price of a house take to grow by a third of its original value if it grows on a
   2.1.1 straight line method (simple interest plan) at a rate of 4,5% per annum?
   2.1.2 Compound growth compounded annually at a rate of 4,5%?
   2.1.3 Which method will you recommend for a bank to offer to its clients and why?

2.1.1 $A = P(1 + in)$

$\frac{4}{3} = 1 + 4{,}5n$

$n = \frac{0{,}33}{4{,}5}$

$n = 7{,}33$ years (0,33x12 =4 months)

It will take 7 years and four months to grow to third of original value on a straight-line method.

2.1.2 $A = P\left(1 + \frac{i}{100}\right)^n$

$n = log_{\left(1+\frac{4,5}{100}\right)} 1{,}33$

= 6,45 years (0,45x12 = 5 months)

= 6 years and 5 months

It will take 6 years and 5 months to grow by third of original value on a compounding method.

2.1.3 The bank will encourage its clients to save using straight line method while they save using compounding method because that's how they earn extra cash.

3. Mpho bought a car to the value of R520 000 and Tsheliso bought the house at the same price. The rate at which the assets are changing is 16,5% per annum. Determine

3.1.1 Mpho's car value after 5 years if it decays at 16,5% per annum on a diminishing-balance.

3.1.2 Tsheliso's house value if it grows at 16,5% after 5 years on a compound interest.

Solutions

3.1.1 $A = P(1 - in)$

$A = 520\ 000(1 - 0{,}165\times 5)$

$A = R91\ 000$

3.1.2 $A = P\left(1 + \frac{I}{100}\right)^n$

A$A = 520\ 000\left(1 + \frac{16,5}{100}\right)^5$

A= R1 115 919, 77

Format: Questions, answers and discussion.

THE EFFECT OF DIFFERENT PERIODS OF COMPOUND GROWTH AND DECAY.

[**Duration**: **45 Min]**

Nthabiseng deposited R3 500 into an account. The interest rate for the first 3 years was 5,5% p.a compounded quarterly, 7,2% p.a compounded semi-annually for the next 2 years and 9,2% p.a effective thereafter. Calculate how much he will have saved after 10 years.

Solution

After the first 3 years, the amount accumulated will be:

$A = P\left(1 + \frac{i}{100}\right)^{n}$

$A = 3500\left(1 + \frac{5.5}{400}\right)^{12}$

$A = R4\ 123\ 24^{c}$

The answer to this becomes the present value for the next 2 years when the interest rate has changed. The value at the end of these 2 years will be:

$A = P\left(1 + \frac{i}{100}\right)^{n}$

$A = 4123{,}24\left(1 + \frac{7{,}2}{200}\right)^{8}$

$A = 5471{,}63$

The answer to this becomes the present value for the next 5 years, when the interest rate changes again, this time to 9,2% effective, which means that it is compounded annually.

$A = 5471{,}63\left(1 + \frac{9.2}{100}\right)^{5}$

$A = R8496{,}30\text{c}$

The process could be summed up as:

$$A = P\left(1 + \frac{i}{100}\right)^{n}$$

$$A = 3500\left(1 + \frac{5{,}5}{400}\right)^{12}\left(1 + \frac{7{,}2}{200}\right)^{8}\left(1 + \frac{9{,}2}{100}\right)^{5}$$

$$A = R8\ 496{,}30^{\text{C}}$$

ACTIVITY **[Duration:30 Min]**

4. Dineo invests R5 000 with Habziz bank. She will get an interest rate of 12,5% per annum. After 4 years she cashes her investment and get far less than she expected. She argues that she is supposed to get R8 009 03[c] instead of R7 500 that the bank gave her. What methods of appreciation did the bank and Tseliso use?
   Solution:
   $A = P(1 + in)$
   $A = 5000(1 + 0{,}125 \times 4)$
   $A = R7\ 500$
   $A = 5000(1 + 0{,}125)^4$
   $A = 8\ 009{,}03$

5. Teboho invest R 5 200 in a savings account. The interest rate for the first 3 years is 7,2% p.a. compounded monthly, thereafter the interest rate changed to 8,5% semi-annually for next 2 years. Determine the amount of money that Teboho has in his savings account at the end of year period.
   Solutions:
   $A = P\left(1 + \frac{i}{100}\right)^n$
   $A = 5200\left(1 + \frac{7{,}2}{1200}\right)^{36}\left(1 + \frac{8{,}5}{200}\right)^4$
   $A = R7\ 617{,}89$

6. Dikeledi is saving for college and decides to put her money into a fixed deposit paying 12% per annum compounded semi-annually. She starts saving with R3 500. After 3 years, she deposits another R6 200. A final deposit of R8 000 is made 7 years after the first deposit. How much money is accumulated in the fixed deposit at the end of 12 years.

   Solution:

   $A = P\left(1 + \frac{i}{100}\right)^n$
   $A = 3500\left(1 + \frac{12}{200}\right)^{24} + 6200\left(1 + \frac{12}{200}\right)^{18} + 8000\left(1 + \frac{12}{200}\right)^{10}$
   $A = 46\ 194{,}96$

NOMINAL AND EFFECTIVE INTEREST RATES.

<u>Format: Questions, answers and discussion.</u>

NOMINAL INTEREST RATE **[Duration: 40 Minutes]**

Nominal rate is annual rate which financial institutions quote. This interest rate does not take into consideration the effect of different compounding periods, which are shorter than the annual period. For an example, 15% per annum compounded monthly is a nominal rate. The annual rate is 15% but the interest is compounded monthly. This means that the accumulated value will be higher than the quoted annual rate.

To illustrate:

R5 000 invested for one year at 15% per annum without monthly compounding:

$A = P(1 + i)^n$

$= 5\ 000(1 + 0{,}15)^1$

$= R5\ 035$

R5 000 invested for one year at 7% per annum with monthly compounding:

$A = 5000(1 + \frac{7}{1200})^{12}$

$= R5\ 750\ 00^{c}$

EFFECTIVE INTEREST RATE

Annual interest rates are therefore the equivalent annual rates that yield the same accumulated amount as rates with different compounding periods. Annual effective rates are higher than quoted nominal rates.

If an interest rate is quoted as $\frac{15\%}{12} = 1{,}25\%\ per\ month.$

Then the formula for effective interest rate is

$$1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$$

I_{eff} = effective rate (annual)

I_{nom} = nominal rate

n = number of compounding per year

Example 1

a) Convert a nominal rate of 10% per annum compounded monthly to an annual effective rate.
b) Convert an annual effective rate of 13,5% per annum, to a nominal rate per annum compounded quarterly.

Solutions:

a) $1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$

$$i_{eff} = \left(1 + \frac{0{,}10}{12}\right)^{12} + 1$$

$$i_{eff} = 0{,}1047$$

$$r_{eff} = 10{,}47\%$$

b) $1 + i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n$

$$1 + 0{,}135 = \left(1 + \frac{I_{nom}}{4}\right)^4$$

$$((1 + 0{,}135)^{0{,}25} - 1\,)\times 4 = i_{nom}$$

$$i_{nom} = 0{,}1287$$

$$r = 12{,}87\%\ \textit{per annum compounded quarterly}.$$

Example 2

Tshitso invests R15 000 for 7 years at 14% per annum compounded monthly.

a) Calculate the future value of the investment using the nominal rate.
b) Convert the nominal rate of 14% per annum compounded monthly to the equivalent effective rate annually.
c) Now use the effective rate to show the same accumulated amount will be obtained as when using the nominal rate.

Solutions:

a) $A = 15\ 000\left(1 - \frac{0{,}14}{12}\right)^{7}$

A = R39 740 77

$i_{eff} = \left(1 + \frac{0{,}14}{12}\right)^{12} - 1$

= 0,1493420292

b) $A = 15\ 000(1 + 0{,}1493420292)^{7}$
= R39 740 77[c]

Format: Questions, answers and discussion.

[Duration: **30 min]**

ACTIVITIES

1. Calculate the effective annual interest rate of a nominal rate of 25% per annum compounded quarterly.

Solution:

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^{n} - 1$$
$$i_{eff} = \left(1 + \frac{25}{12}\right)^{12} - 1$$
$$i_{eff} = 02807$$
$$r_{eff} = 28{,}08\%$$

2. Mohau invests R80 000. He is quoted a nominal interest of 8,5% per annum compounded monthly.

2.1.1 Calculate the effective rate per annum correct to THREE decimal places.
2.1.2 Use the effective rate to calculate the value of Mohau’s investment if he invested the money for 6 years.
2.1.3 Suppose Mohau invests his money for a total 5 years, but after two years (24 months) makes a withdrawal of R 45 000, how much will he receive at the end of the 5 years.

Solution:
2.1.1

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^{n} - 1$$
$$i_{eff} = \left(1 + \frac{0{,}085}{12}\right)^{12} - 1$$
$$i_{eff} = 0{,}08839$$

$r_{eff} = 8{,}84\%$
2.1.2
2.1.3

3. Find the nominal interest rate per annum compounded quarterly that is equivalent to an effective interest rate of 13,5% p.a. Give your answer correct to 2 decimal places.

Format: Questions, answers and discussion.
THE PRESENT VALUE ANNUITIES **[Duration: 40 min]**

A reducing balance loan is often referred to as **present value annuity.** In a present value annuity, a sum of money is normally borrowed from a financial institution and paid back with interest by means of regular payments at equal intervals over a time period. The loan is paid off when it together with interest charges is paid off.

Example

Suppose that a loan is repaid by means of a payment of R500 one month after the loan was granted and one further payment of R500 one month after the first payment of R500. Calculate the amount borrowed (the present value of the loan). The interest rate is 8% per annum compounded quarterly.

The present value at T_0 of the payment at T_1 can be calculated as follows:

Method 1:

$A = 500\left(1+\frac{0{,}08}{12}\right)^{-1}$ = R496,69

The present value at T_0 of the payment at T_2 can be calculated as follows

$\mathrm{A} = 500\left(1+\frac{0{,}08}{12}\right)^{-1}$ = R493,40

The amount borrowed is called the loan (P) is the sum of the two present values.

$$A = 500\left(1+\frac{0{,}08}{12}\right)^{-1}\left(1+\frac{0{,}08}{12}\right)^{-2} = R990{,}09$$

Method 2

$$A = \frac{x(1-(1+i)^{-n})}{i}$$

$x - number\ of\ payments\ made\ per\ period.$

i-interest rate as a decimal $=\frac{r}{100}$

n- number of payments made

Using the present value calculated using the formula:

$P = \frac{x[1-(1+I)^{-n}]}{i}$ where:

$P = \frac{500\left[1-\left(1+\frac{0,08}{12}\right)^{-2}\right]}{\frac{0,06}{12}}$ = R990,09

ACTIVITY

1. Thabo took out a home loan of R650 000 at an interest rate of 18% per annum compounded monthly. He plans to repay this loan over 20 years and his first payment is made one month after the loan is granted.

   Calculate the value of Thabo’s monthly instalment.

   Mpuse took out a loan for the same amount and the same interest rate as Thabo. Mpuse decided to pay R11 500 at the end of every month. Calculate how many months it took Mpuse to settle the loan.

   Who pays more interest, Thabo or Mpuse? Justify your answer.

2. Nontombi is planning to buy her first home. The bank will allow her to Nontombi to use 33% of her monthly salary to repay the bond.

   2.1 Calculate the maximum amount that the bank will allow Nontombi to spend each month on her bond repayments, if she earns R28 350 per month.

   2.2 Suppose at the end of each month, Nontombi repays the maximum amount allowed by the bank. How much money does Nontombi borrow if she takes 30 years to repay the loan at a rate of 9,5% p.a, compounded monthly? (The first payment is made one month after the loan is granted.)

3. Thabo bought a house for R1 280 000. He paid a deposit of 15% of the selling price of the house. He obtained a loan from the bank at an interest rate of 10,5% per annum, compounded monthly, to pay the balance of the selling price. He agreed to pay monthly instalments of R12 000 on the loan.

   3.1.1 How much money did Thabo borrow from the bank?

   3.1.2 How many months will it take to repay the loan?

   3.1.3 Calculate the balance of his loan immediately after his 90^{th} instalment.

3.1.4 Thabo experienced financial difficulties after the 90^{th} instalment and did not pay the 91^{st} to the 95^{th} instalment. At the end of the 96^{th} month he increased his monthly instalment to pay off the loan in the same time interval as planned initially. Calculate the value of his new monthly instalment.

FUTURE VALUE ANNUITY

Dimpho wants to invest in a unit trust. She deposits R620 immediately in to the trust and thereafter makes a monthly payment at the end of each month into the trust. Interest is sat at 8,25% per annum compounded monthly. Calculate the value in the trust at the end of 7 years.

Solution:

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$F = \frac{620\left[\left(1+\frac{8,25}{1200}\right)^{85} - 1\right]}{\frac{8,25}{1200}}$$

$F = R71\ 269\ ,43$ c

ACTIVITIES

1. Lerato wants to save R850 000. She opens an investment account with a bank which will pay an interest of 6,25% p.a, compounded monthly. She intends to deposit a monthly of R1 600 into the account for 12 years.
   1.1 Determine whether Lerato will save enough in 12 years to make R850 000.
   1.2 How much must Lerato deposit every month to accumulate R850 000 in12 years.

Solutions:

1.1

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$F = \frac{1600\left[\left(1+\frac{0,0625}{12}\right)^{144}\right]}{\frac{0,0625}{12}}$$

$F = R341\ 877\ ,83$

Lerato's monthly savings will not meet her target of R850 000.

1.2

$$850\ 000 = \frac{x\left[\left(1+\frac{0,0625}{12}\right)^{144} - 1\right]}{\frac{0,0625}{12}}$$

$x = R3\ 978,03$

2. A business buys a machine that costs R150 000. The value of the machine depreciates at 9% per annum according to the reducing –balance method.

   2.1.1 Determine the scrap value of the machine at the end of 6 years.

   2.1.2 After 6 years the machine needs to be replaced. During this time the inflation remained constant at 6,5% per annum. Determine the cost of the new machine at the end of 6 years.

   2.1.3 The business estimates that it will need R 1400 000 by the end of 6 years. A sinking fund for R140 000, into which equal monthly instalments must be paid, is set up. Interest on this fund is 8,25% per annum, compounded monthly. The first payment will be immediately and the last payment will be made at the end of the 6- year period. Calculate the value of the monthly payments into the sinking funds.

   Solutions:

   2.1.1

   $A = P(1-i)^n$

   $= 150\ 000\left(1 - \frac{9}{100}\right)^6$

   $= R\ 85\ 180,39$

   2.1.2 $A = P(1+i)^n$

   $= 150\ 000(1 + 0,0825)^6$

   $= R218\ 871,35$

   2.1.3

   $$F = \frac{x[(1+i)^n - 1]}{i}$$

   $$140000 = \frac{x\left[\left(1+\frac{8,25}{1200}\right)^{72} - 1\right]}{\frac{8,25}{1200}}$$

   $x = R1\ 509,28$

3. Disebo receives R1 250 000 upon her retirement. She invests this amount immediately at an interest rate of 10,5% per annum, compounded monthly.
   She needs an amount of R17 500 per month to maintain her current lifestyle. She plans to withdraw the first amount at the end of the first month. For how long will she be able to live from her investment?

   Solution:

   $$P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$1250000 = \frac{17500\left[1 - \left(1 + \frac{10,5}{1200}\right)^{-}\right]}{\frac{10,5}{1200}}$$

MIXED QUESTIONS

1.1 Samuel invested an amount with ABC bank at an interest 12% p.a. compounded monthly. His investment grew to R8450 at the end of 10 years. Determine the amount that Samuel initially invested.

1.2 If the inflation rate remains at a constant 4,7 % p.a., what amount of time will it take for a certain amount to be worth half of the original amount.

1.3 Lebogo buys a tractor for Rx. She plans to replace this tractor after 5 years.
The tractor depreciates by 20% p.a. according to the reducing balance method.
The price of a new tractor is expected to increase by 18% p.a.
She calculates that if she deposits R8 000 into a sinking fund at the end of each month, it would exactly provide for the shortfall 5 years from now when she should pay for the new tractor. The bank offers 10% p.a. interest compounded monthly.

1.3.1 Calculate the scrap value of the tractor after 5 years, in terms of x?

1.3.2 Determine the price of the new tractor after 5 years, in terms of x?

1.3.3 Calculate the amount accumulated in the sinking fund after 5 years.

1.3.4 Determine the value of x, the price of the original tractor. .

2. Thabo bought a house for R980 000. He paid a deposit of 10% of the selling price of the house. He obtained a loan from the bank at an interest rate of 11% per annum, compounded monthly, to pay the balance of the selling price. He agreed to pay monthly instalments of R10 000 on the loan.

2.1 How much money did Thabo borrow from the bank?

2.2 How many months will it take to repay the loan?

2.3 Calculate the balance of his loan immediately after his 90th instalment.

2.4 Thabo experienced financial difficulties after the 90th instalment and did not pay the 91st to the 95th instalment. At the end of the 96th month he increased his monthly instalment to pay off the loan in the same time interval as planned initially. Calculate the value of his new monthly instalment.

3. A business installs a server for R500 000. The value of the server depreciates at 20% per annum according to the diminishing-balance method.

3.1. Calculate the scrap value of the server at the end of 6 years.

3.2 The server needs to be replaced after 6 years. Calculate the cost of the new server if the inflation rate is at 7% per annum. The older server will be traded in.

3.3. On the day, the server gets installed, the business sets up a sinking fund into which equal monthly instalments must be paid. Interest on this fund is 8% per annum compounded monthly. The first payment will be made immediately and the last payment will be made at the end of the 6-year period. Calculate the value of the monthly instalment into the sinking fund.

3.4 The business decides to rather pay a monthly instalment of into the sinking fund. After how many months will there be more than R1 000 000 in the fund?

4.1 The estimated inflation rate is an average of 3% p.a. How much will R800 000 today be worth 10 years from now?

4.2 Peter saw an advert of a house worth R500 000 at an upmarket area.

He paid 20% deposit cash and took a loan from the bank to pay for the balance of

the cost of the house.

4.2.1 What amount was the loan that Peter took from the bank?

4.2.2 He planned to pay the loan back over 20 years starting from a month after

the loan was taken. The bank offers loans at 12, 5% interest p.a.

compounded monthly. If the loan is R400 000,

calculate the value of his monthly instalments.

4.2.3 After 10 years he won lotto and wants to pay off the house.

How much of the lotto winnings did he use to pay off the house?

4.2.4 How much has he paid for the house when the debt is settled

after 10 years?

5.1 Thabo bought a car for R135 000, it depreciates annually at a compound rate. After 6 years it is worth R55 000. At what rate did the value of a car depreciate?

5.2 Mpho sold his property for R4 150 000 and invested the money at 9,5% p.a., compounded quarterly.

Five years later he used the interest gained on the investment to buy another property for R 920 000 and obtained a mortgage bond for the remaining amount. The bond was granted for 15 years at 8,25% interest p.a. compounded monthly.

5.2.1 Calculate interest Mpho on the investment for first five 5 years.
5.2.2 Determine the value of the bond that Mpho obtained.

5.2.3 Calculate the monthly payment that he should make to pay off the bond.
5.2.4 He paid off the bond in 15 years. How much interest did he pay on the bond?

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